Synchronization of Kuramoto oscillators in scale-free networks

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Abstract. – In this work, we study the synchronization of coupled phase oscillators on the underlying topology of scale-free networks. In particular, we assume that each network's component is an oscillator and that each interacts with the others following the Kuramoto model. We then study the onset of global phase synchronization and fully characterize the system's dynamics. We also found that the resynchronization time of a perturbed node decays as a power law of its connectivity, providing a simple analytical explanation to this interesting behavior.

The behavior of an isolated generic dynamical system in the long-term limit could be described by stable fixed points, limit cycles or chaotic attractors [1]. We have also learned in recent years that when many of such dynamical systems are coupled together, new collective phenomena emerge. In this way, the study of regular networks of dynamical systems has shed light on a number of natural phenomena ranging from earthquakes to ecosystems and living organisms [2–4]. One of the most fascinating phenomena in the behavior of complex dynamical systems made up of many elements is the spontaneous emergence of order and the phenomenon of collective synchronization [5], where a large number of the system's constituents forms a common dynamical pattern, despite the intrinsic differences in their individual dynamics. Of recent interest are a plenty of biological examples that have become accessible at least numerically with the advent of modern computers [6].

On the other hand, it has been recently shown that many biological [7,8], social [9], and technological [10] systems exhibit an intricate pattern of interconnections in the form of complex networks [6]. This structural complexity cannot be described by the couplings of a regular network. In order to characterize topologically these complex networks, one computes the probability, $P(k)$, that any given element of the network has k connections to other nodes. Interestingly, many real-world networks such as the Internet, protein interaction networks and social webs [11] can be well approximated by a power law connectivity distribution, $P(k) \sim k^{-\gamma}$. Additionally, they are characterized by the existence of key nodes which drastically reduce the average distance between nodes, the so-called small-world property [12].

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Most of the studies performed so far have scrutinized the structure of complex networks and studied prototype models ran on top of these networks [13]. Their peculiar topological properties have been shown to lead to radical changes when dynamical processes such as epidemic spreading and percolation phenomena are studied on top of complex heterogeneous networks [14–17]. However, for biological and other applications, it would be relevant to consider the nodes of a given network as dynamical systems with own identity. Examples of such dynamical systems are ensembles of coupled and pulse-couple oscillators with and without time delay, widely used because of their relevance to natural systems such as chirping crickets and flashing fireflies, among others [18, 19]. Besides, there are several studies where the conditions for complete synchronization in complex networks are scrutinized [20–22].

In this paper, we numerically study the synchronization of coupled phase oscillators following Kuramoto's model [23, 24] on the underlying topology of a scale-free (SF) network. We report on the system dynamics by computing the conventional order parameter. The onset of synchronization is found at a nonzero value of the coupling strength. Remarkably, the transition from a desynchronized to a synchronous state can be characterized by the mean-field exponent found for globally coupled oscillators. The heterogeneous character of the network allows us to explore the robustness of the synchronized state under a single perturbation as a function of the connectivity of the perturbed oscillator. Interestingly, we found that the more connected a node is, the more stable it is.

Let us consider an SF network where each node i $(i = 1, \cdots, N)$ is a planar rotor characterized by an angular phase, θ_i, and a natural or intrinsic frequency ω_i. Two oscillators interact if they are linked together by an edge of the underlying network. The individual dynamics of the i-th node is described by

$$\frac{d\theta_i}{dt} = \omega_i + \lambda \sum_{j \in I(i)} \sin\left(\theta_j - \theta_i\right), \tag{1}$$

where $I(i)$ is the set of neighbors of the rotor i as dictated by the architecture of the network and λ is the coupling strength, identical for all edges. The natural frequencies and the initial values of θ_i are randomly drawn from a uniform distribution in the interval $(-1/2, 1/2)$ and $(-\pi, \pi)$, respectively. On the other hand, in order to produce SF networks we have used the BA procedure [25]. In this model, starting from a set of m_0 nodes, one preferentially attaches at each time step a newly introduced node to m older ones ($m = 3$ has been set). The procedure is repeated many times and a network with a power law degree distribution $P(k) \sim k^{-\gamma}$ with $\gamma = 3$ and average connectivity $\langle k \rangle = 2m = 6$ builds up. This network is a clear example of a highly heterogeneous network because the degree distribution has unbounded fluctuations when $N \to \infty$.

The original Kuramoto model corresponds to the simplest case of globally coupled (all-to-all), equally weighted oscillators where the coupling strength $\lambda = K/N$ to ensure that the model is well behaved in the thermodynamic limit [23, 24]. The onset of synchronization occurs at a critical value of the coupling strength, $K_c = 2/\pi g(\omega_0)$, where $g(\omega)$ is the distribution from which the natural frequencies are drawn evaluated at the mean frequency ω_0. The second-order phase transition is characterized by the following order parameter:

$$r(t) = \left| \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)} \right| \tag{2}$$

which behaves when both $N \to \infty$ and $t \to \infty$ as $r \sim (K - K_c)^\beta$ for $K \geq K_c$ being $\beta = 1/2$.
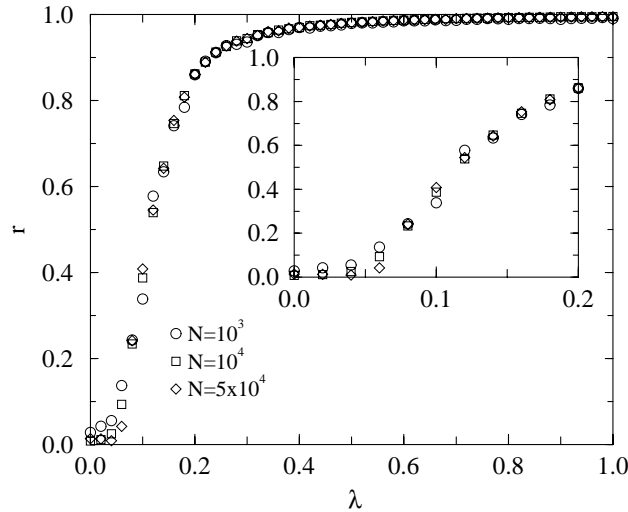
Fig. 1 – Coherence r as a function of λ for several system sizes. The onset of synchronization occurs at a critical value $\lambda_c = 0.05(1)$. Each value of r is the result of at least 10 network realizations and 1000 iterations for each N. The inset is a zoom around λ_c.

In order to inspect the dynamics of the N oscillators on top of complex topologies, we have performed extensive numerical simulations of the model in BA networks. Starting from $\lambda = 0$, we increase at small intervals its value. Then, we integrate the equations of motion, eq. (1), over a sufficiently large period of time (at least 10^4 integration steps) to ensure that the system is in a stationary state, and the order parameter r is computed. The procedure is repeated gradually increasing λ until the system evolves to a state of collective phase synchronization.

In the case of random SF networks, the global dynamics of the system is qualitatively the same as for the original Kuramoto model as shown in fig. 1 for several system sizes. As the coupling is increased from small values, the strength of the interactions is not enough to break the incoherence produced by their individual dynamics. This behavior persists until a certain critical value λ_c is crossed. At this point, some elements lock their relative phase and a cluster of synchronized nodes comes up. This constitutes the onset of synchronization. Beyond this value, the population of oscillators is split into a partially synchronized state made up of oscillators locked in phase that adds to r and a group of nodes whose natural frequencies are too spread as to be part of the coherent pack. Finally, after further increasing the value of λ, more and more nodes get entrained around the mean phase and the system settles in a completely synchronized state where $r \approx 1$.

This picture is clearly illustrated in fig. 2, where we have plotted the distributions of phases $D(\theta)$ for five different values of λ between 0 and 1. Below the critical point the phases are uniformly scattered through the entire interval $(-\pi, \pi)$. When $\lambda > \lambda_c$, the distribution shrinks around the mean value $\theta = 0$ and the dispersion gets smaller as λ grows, signaling that the system is in a synchronous state.

Now we proceed to investigate the critical parameters of the system dynamics. First, we should determine with precision the exact value of λ_c. This is not an easy task because there are several sources of heterogeneity and averages should be properly taken —from network realizations to the initial distributions of θ_i and ω_i— through lengthly numerical calculations. Additionally, finite-size effects come into play. We have determined λ_c and studied the critical
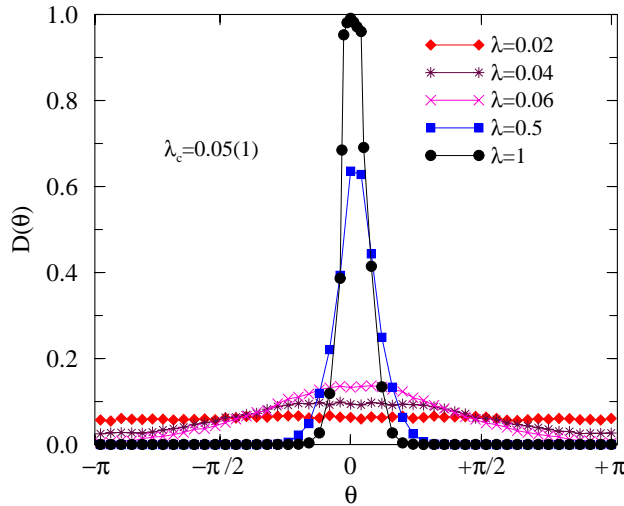
Fig. 2 – Normalized phase distributions $D(\theta)$ for different values of the control parameter λ. The curves depicted correspond to values of λ below, near and above λ_c as indicated. The network is made up of $N = 10^4$ nodes.

behavior near the synchronization transition by means of a standard finite-size scaling (FSS) analysis [26]. For a given network size N, we have that no synchronization is attained below λ_c and that $r(t)$ decays to a small residual value of size $O(1/\sqrt{N})$. Hence, the critical point may be found by examining the N-dependence of $r(\lambda, N)$. For $\lambda < \lambda_c$ (sub-critical regime), the stationary value of r falls off as $N^{-1/2}$, while for $\lambda > \lambda_c$, r approaches a nonzero, stationary value as $N \to \infty$ though still with $O(1/\sqrt{N})$ fluctuations. In this way, plots of r vs. N as that of fig. 3, allow us to locate the critical point λ_c.

Following the FSS procedure, our best estimate gave a value for the critical coupling strength $\lambda_c = 0.05(1)$. Besides, we found that $r \sim (\lambda - \lambda_c)^{\beta}$ above the critical point with $\beta = 0.46(2)$ indicating that the square-root behavior typical of the mean-field version of the model (all-to-all architecture) also holds for SF networks. In addition, it is worth stressing the very existence of a critical point. This is the opposite to what has been found when percolation or epidemic spreading models are ran on top of complex heterogeneous networks even if correlations are taken into account [14–17, 27]. Furthermore, the critical point shifts to the left when the average connectivity $\langle k \rangle$ of the underlying network increases, but it is always distinctly different from zero. Other numerical calculations (not shown) indicate that the relation $\lambda_c^{\langle k \rangle} \cdot \langle k \rangle$ is roughly constant when $\langle k \rangle$ varies, which implies that there is a nontrivial critical point even in the infinite size limit [28].

Once we have characterized the emergence of spontaneous synchronization, we look at the stability and robustness of the synchronous state. The most interesting and influential topological property of complex heterogeneous networks is that the fluctuations of the connectivity distribution are unbounded. In other words, there is a clear distinction between the nodes according to their connectivities. From a practical point of view —and worth taking into account as a design principle in natural or artificial networks— it would be particularly relevant that the most highly connected nodes were also the most robust with respect to synchronization when they were perturbed. We have computed the average time $\langle \tau \rangle$ it takes for a node to be again in the synchronized cluster as a function of its connectivity k after being perturbed
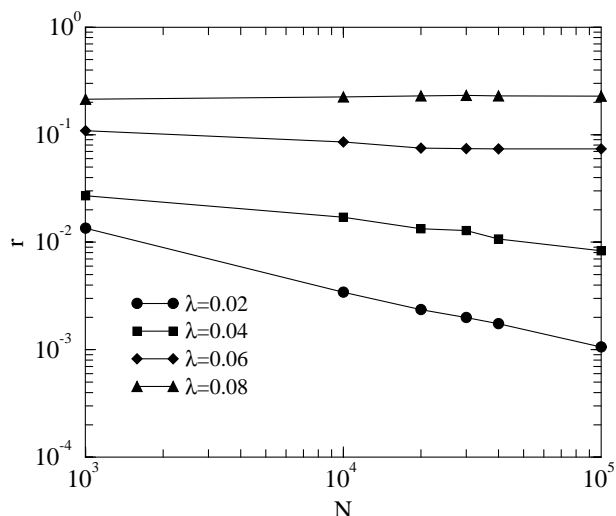
Fig. 3 – Finite-size scaling analysis used to determine the value of the critical point. The curves suggest the existence of a critical point above 0.04. The values of r are collected after 10^4 integration steps and averaged over a time window of 100 additional integration steps. Finally, each point is averaged over at least 10 network realizations and 10^3 different initial conditions.

and put out of synchronization. This is done by assigning to a randomly chosen node i of connectivity k_i a new phase θ_i which differs in $-\pi$ to its synchronization value. In this way, all nodes are perturbed in the same amount and one may calculate the time it takes for i to recover from the perturbation. Note that as λ is high enough, the oscillator i ends up in the
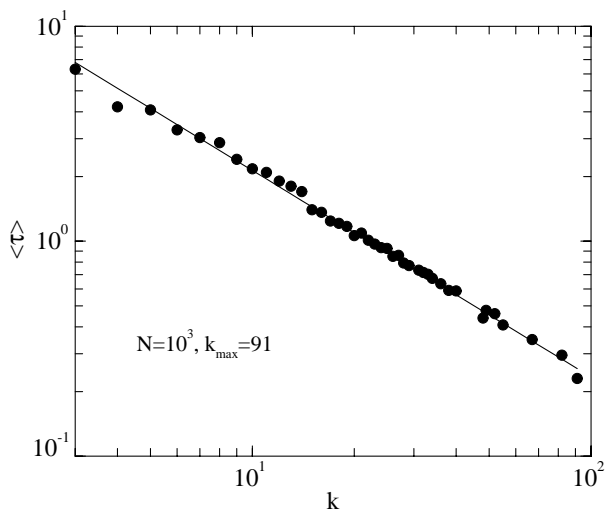

Fig. 4 – Log-Log plot of the dimensionless average time $\langle \tau \rangle$ it takes for a node of connectivity k to be back in the synchronous state after being perturbed. The least-square fit to the data gives for the exponent $\nu = 0.96(1)$. The results were averaged over 10 network realizations and 500 perturbations for each k. λ is set to 0.4. See the text for details on the definition of τ.

synchronous state with the same θ_i it had before being perturbed. The results are drawn in fig. 4, where a clear power law dependency $\langle\tau\rangle \sim k^{-\nu}$, with $\nu = 0.96(1)$, can be identified. Hence, the more connected a node is, the more stable it is. The power law behavior points to an interesting result, namely, it is more easy for an element with high k to get locked in phase with its neighbors than for a node linked to just a few others. Furthermore, the destabilization of a hub does not destroy the synchrony of the group it belongs to. Instead, it works the other way around, the group formed by the hub's neighbors recruits it again.

This behavior and the dependency $\langle\tau\rangle \sim k^{-1}$ may be understood by the following simple argument. As we are perturbing a single node i and this perturbation, ξ_i, is small, we can assume that it affects only the first neighbors of the perturbed node. Hence, the stability analysis can be locally reduced to the problem of how such a perturbation relaxes in a star topology (a single perturbed hub attached to $k \gg 1$ oscillators). This approximation is particularly suitable in random networks with arbitrary degree distributions (like the BA one) because the probability of finding loops (tringles, cycles, etc.) is small and vanishes as N grows. Linearization of eq. (1) for this configuration leads to $\xi_i^{\eta} = \xi_i^{\eta}(0)e^{\eta_i t}$, where η_i is the eigenvalue corresponding to the oscillator i. Henceforth, the times $\langle\tau\rangle$'s are given by the inverse of the eigenvalues, which for a star configuration are $\eta_i = -1$, for $i = 1, \cdots, N - 2$ and $\eta_{N-1} = \eta_{\text{hub}} = -N = -k_{\text{hub}} - 1$ [29]. In other words, the fastest relaxation rate in a start topology corresponds to the hub and goes like $1/k_{\text{hub}}$ for $k_{\text{hub}} \gg 1$, while the rest of the oscillators all have the same relaxation times. Additionally, the eigenvector associated to the eigenvalue of the hub indicates that it moves a lot while the other nodes change very little. Finally, if we superpose the effects of many perturbations to different nodes (each one being a $k_{\text{hub}} + 1$ star), it comes out that each of them contributes with $1/k_{\text{hub}}$, $k_{\text{hub}} = k_{\text{min}}, \cdots, k_{\text{max}}$ to the $\langle\tau\rangle$-dependency on k, leading to the law $\langle\tau\rangle \sim k^{-1}$ depicted in fig. 4.

In summary, we have studied the synchronization of Kuramoto oscillators on top of complex scale-free networks. We have found that the onset of synchronization occurs at a nonzero value, though small, with a critical exponent around 0.5. We also found that when the synchronous state has been attained, highly connected nodes are more robust under perturbations and they are recovered in a time which depends on their degree as a power law with exponent close to -1. This law support the suggestion that the actual topology of scale-free networks may be a result of some kind of optimization mechanism at a local scale, optimizing in this case the fitness for synchronization of highly connected nodes. This question as well as the influence of time delay and noise on the synchronization of complex networks make it necessary to study in more detail the connection between graph theory, dynamics on networks and nonlinearity in future modeling of complex networks.

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