

Robustness of cooperation in the evolutionary prisoner's dilemma on complex networks

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New Journal of Physics **9** (2007) 184

Received 19 February 2007

Published 28 June 2007

Online at <http://www.njp.org/>

doi:10.1088/1367-2630/9/6/184

Abstract. Recent studies on the evolutionary dynamics of the prisoner's dilemma game in scale-free networks have demonstrated that the heterogeneity of the network interconnections enhances the evolutionary success of cooperation. In this paper we address the issue of how the characterization of the asymptotic states of the evolutionary dynamics depends on the initial concentration of cooperators. We find that the measure and the connectedness properties of the set of nodes where cooperation reaches fixation is largely independent of initial conditions, in contrast with the behaviour of both the set of nodes where defection is fixed, and the fluctuating nodes. We also check for the robustness of these results when varying the degree heterogeneity along a one-parametric family of networks interpolating between the class of Erdős–Renyi graphs and the Barabási–Albert networks.

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1. Introduction

Evolutionary dynamics has proved to be a useful theory to describe evolution of biological systems at all levels of organization [1]. Rooted in the basic tenet of Darwinism, the replicator dynamics [2]–[4] of evolutionary game theory provides an elegant mathematical description of how natural selection among (phenotypes) strategies takes place when the reproductive success of individuals (and then the future abundance, i.e. frequency, of strategies) depends on the current phenotypic composition of the population (frequency-dependent fitness). In this regard, one of the current theoretical challenges to the explanatory powers of evolutionary game dynamics is the understanding of the observed evolutionary survival of cooperative behaviour among individuals when selfish actions provide a higher benefit (fitness). Perhaps the best suited (and most used) model to formally describe the puzzle of how cooperation arises is the prisoner’s dilemma (PD), a two-players-two-strategies game, where each player chooses one of the two available strategies, cooperation or defection: a cooperator receives R when playing with a cooperator, and S when playing with a defector, while a defector earns P when playing with a defector, and T against a cooperator. When $T > R > P > S$, the game is a PD. Given this payoff ordering, in a well-mixed (unstructured) population where each agent interacts with all other agents (or a representative sample of the population composition), defectors are fitter and thus the fraction of cooperators asymptotically vanishes.

Among the various mechanisms that have been proposed to explain how natural selection can lead to cooperative behaviour (like kin selection, group selection, direct or indirect reciprocity) [5], a simple one is based on leaving off the well-mixed population hypothesis, so that each individual only interacts with agents in its neighbourhood, as specified by some graph or network of ‘social’ interactions. Agent-based-modelling approaches [6] of this kind in theoretical biology [7], economics [8] and social sciences [9] often benefit in a natural way from statistical physics methods, concepts and techniques (also scientists), so favouring fruitful (synergic) interdisciplinary (socio-, bio-, econo-)physics research [10], often termed the physics of complex systems [11, 12].

Early pioneering numerical work [13] on the PD game in two-dimensional square lattices, made the observation that, unlike in unstructured populations, cooperators and defectors can coexist in the lattice indefinitely. In [13] each individual node played with its immediate neighbours each time step accumulating a payoff, then updated its strategy by imitating the one of highest payoff in its neighbourhood, including itself (best-takes-over reproduction rule) and back again for very large times. When passing from a ‘mean field’ (well mixed population)

interaction description to a lattice structure of interactions, one has to specify various details (of varying importance) on both, (i) the lattice characteristics, e.g. regular or not, randomness of various kinds, finite size effects, etc, and (ii) the specific form of the microscopic dynamics of reproduction process, e.g. deterministic rules or probabilistic ones, synchronous or asynchronous updating, what types of stochastic fluctuations are allowed, etc. The study of many, if not most, of the important aspects of the issue have generated for more than a decade a wealthy literature, of a great interest from the statistical physics perspective (e.g. [14]–[28]; for a recent review, see [10], where an extensive list of references can be found).

Nowadays, the existence of *cooperation-promoting feedback mechanisms that are rooted deep into the interaction structure* is indisputably accepted. It has been termed *spatial*, or *lattice reciprocity*, in analogy to *direct* reciprocity (through iterated game strategies) and *indirect* reciprocity (through reputation, or scoring, of agents). Simply said, the clustering of cooperators in the lattice could provide high enough fitness to the cooperator nodes exposed to invasion, to the extent of preserving cooperators from evolutionary extinction, even when defection is blatantly favoured by the one-shot (two-players) game analysis. For negligible values of $P - S \simeq 0$, when $T - R$ increases from zero cooperation decreases slowly, and becomes zero at values of $(T/R) - 1$ well beyond zero. The region (in parameter space) of coexistence of strategists is the genuine battlefield where the competition between strategies adopts interesting, nontrivial aspects: the transition region between two clear-cut phases, i.e. all-cooperators (all-C) prevailing at $T/R \simeq 1$, and all-defectors (all-D) at higher values of T/R . More recently, a set of works have extended this perspective to a most intriguing and ubiquitous class of networks, say scale-free (SF) networks, a ‘focus issue’ nowadays.

There is an accumulated evidence that many real biological [29, 30], social [31] and technological [32]–[34] systems are neither regular nor simplest random graphs (not to say well-mixed populations) of entities or agents, but they are described by some distinctive metric (path length based) and topological (structure and size of local neighbourhoods) properties. They often show a so-called SF distribution density of degree, $P(k) \sim k^{-\gamma}$, where the degree k of a node is the number of connections it shares with its neighbours [35, 36], so their connectivity patterns depart considerably from lattice homogeneity (lacking of a sharp characteristic scale of connectivity). The ubiquity and importance of complex networks raised quite naturally the question of how natural selection works on top of different types of complex networks of agents [19], [37]–[41]. In this case (as in other nonlinear dynamical processes in networks [42, 43]) one has to deal with two sources of complexity, the evolutionary dynamics and the complex structure of the substrate, which are entangled. Interestingly, the sort of processes that evolutionary game dynamics is aimed to model may well be very relevant to understand real networked systems through the study of a variety of scenarios of co-evolution of both strategies (phenotype survival) and network (evolving topological features) [38]. Among other works exploring various aspects on the evolution in complex networks, see [44]–[46]. From here we focus attention on fixed network settings and how degree heterogeneity influences evolutionary dynamics of PD.

Some recent extensive numerical works on PD (and closely related) games [39]–[41] on SF networks, using probabilistic updating rule (random neighbour pair-comparison, and update with probability proportional to fitness difference) have shown that the absence of a sharp characteristic scale of degree in the network greatly enhances the ‘lattice reciprocity’ mechanisms of evolutionary survival of cooperation. For example, highly connected (hubs) cooperator nodes have the chance of high payoffs and resist invasion well by easily invading less connected neighbours, which in turn increase the hub’s payoffs and invading capabilities [40]; this positive

feedback mechanism does not operate in the case of defector hubs and illustrates in a simple way one of the biasing effects of graph heterogeneity.

In a recent exploration of these heterogeneity-based cooperation-promoting mechanisms, using the kind of implementation of replicator dynamics on graphs specified above in the previous paragraph, one observes generically [41] that fixation of the cooperation (as well as defection) strategy on certain nodes occurs after (often-not-large) sensible transients, so that any asymptotic trajectory of population states defines a partition of the network into three sets: the set \mathcal{C} of nodes where cooperation is fixed, the set \mathcal{D} of nodes where defection is fixed, and the set \mathcal{F} of fluctuating nodes that experience forever cycles of invasion by the competing strategies. In other words, the observed stationary value of the average fraction \bar{c} of cooperators (see definition in section 3), in any asymptotic (long-term) trajectory, has two additive contributions: (i) the relative size $\mu(\mathcal{C})$ of the set of pure cooperators, and (ii) the overall fraction of time \bar{T}_c spent by fluctuating nodes as cooperators, weighted by its relative size $\mu(\mathcal{F})$, say

$$\bar{c} = \mu(\mathcal{C}) + \mu(\mathcal{F})\bar{T}_c. \quad (1)$$

The analysis of global connectedness inside the sets \mathcal{C} and \mathcal{D} of fixed strategy nodes reveals that the lack of a significant characteristic scale of degree is neatly associated to a simply connected \mathcal{C} set, while \mathcal{D} is fragmented into many clusters in the wide transition region (coexistence of strategies) between asymptotic uniform ($\mu(\mathcal{C}) = 1$, all-C, and $\mu(\mathcal{D}) = 1$, all-D) equilibria. This structure of \mathcal{D} in the strategies coexistence regime is similar to that exhibited by both \mathcal{C} and \mathcal{D} sets for the Erdős–Renyi (ER) random class of networks (i.e. with Poissonian distribution density of degrees, and thus a significant characteristic scale: the network average degree) [41]. All previous results [41], were obtained for a unbiased (50%) initial proportion of (randomly placed) cooperators, for all the analysed stochastic trajectories.

In this paper, we are interested in exploring the robustness of these observations reported in [41] on the behaviour of the partition sets, for a limiting one-parameter form of the PD game, say $P - S = 0$: the border with the snowdrift game (see next section). Robustness against parameter $P - S$ variation, and others, will be analysed elsewhere [47]. In particular, we focus here on two aspects of robustness: firstly, the influence of varying initial fraction of cooperators on the network partition sets (\mathcal{C} , \mathcal{D} , \mathcal{F}) of asymptotic trajectories. The model, its dynamical rules and structural characteristics, as well as the necessary technical details, are the contents of section 2. The results are described and analysed in section 3. Secondly, in section 4, we show how those observed behaviours of the partition vary along an interpolating family of networks whose heterogeneity can be one-parametric tuned, from the ER limit to the Barabasi–Albert (BA) limit, that is, we check robustness against decreasing heterogeneity of the network. Conclusions and some prospective remarks can be found in section 5.

2. The model

The PD game is defined in its more general form by the payoff matrix:

$$\begin{pmatrix} R & S \\ T & P \end{pmatrix}, \quad (2)$$

where the element a_{ij} is the payoff received by an i -strategist when playing against a j -strategist, with $i = 1$ meaning cooperator, and $i = 2$ defector. The payoff ordering is given by $T > R > P > S$. Other payoff orderings have received other names, e.g. $T > R > S > P$ corresponds to the so-called snowdrift (or hawks and doves, or chicken) game. Following several studies [13, 39], the PD payoffs have been set to $R = 1$ (so the reward for cooperating fixes the payoff scale), $T = b > 1$, $P = 0$ (no benefit under mutual defection), and $P - S = \epsilon = 0$. This last choice places us at the very frontier of the PD game. It has the effect of not favouring any strategy when playing against defectors (while being advantageous to play defection against cooperators). Small positive values of the parameter $\epsilon \ll 1$ leads to no qualitative differences in the results [13, 39, 49], so the limit $\epsilon \rightarrow 0^+$ is agreed to be continuous.

The dynamic rule is specified as follows: each time step is thought of as one generation of the discrete evolutionary time, where every node i of the system plays with its nearest neighbours and accumulates the payoffs obtained during the round, say P_i . Then, individuals are allowed to synchronously change their strategies by comparing the payoffs they accumulated in the previous generation with that of a neighbour j chosen at random. If $P_i > P_j$, player i keeps the same strategy for the next time step, when it will play again with all of its neighbourhood. On the contrary, whenever $P_j > P_i$, i adopts the strategy of j with probability $\Pi_{i \rightarrow j} = \beta(P_j - P_i)$, where $\beta^{-1} = \max\{k_i, k_j\}b$. Note that this dynamic rule, though stochastic, does not allow the adoption of irrational strategy, i.e. $\Pi_{i \rightarrow j} = 0$ whenever $P_j \leq P_i$.

Let us now specify precisely the family of networks on top of which the evolutionary PD game is evolved. Strategists are located on the vertices of a fixed graph of average connectivity $\langle k \rangle = 4$. The heterogeneity of the networks is controlled by tuning a single parameter α , so that when $\alpha = 0$ the networks are of the ER class of random graphs, and when $\alpha = 1$ they are of the BA [48] SF networks class. Let us first describe the algorithm to construct a BA network of size N . In this case, one starts from a fully connected set of m_0 nodes and at each time step a new node is linked to $m = 2$ nodes preferentially chosen, namely, the probability that node i receives one new link is proportional to its degree, $k_i / \sum_j k_j$. Avoiding multiple connections and iterating the preferential attachment rule $N - m_0$ times a SF network with an exponent $\gamma = 3$ is generated. On the other hand, random single-scale networks are built up following the standard recipe to generate ER networks [36]. Finally, networks with an intermediate degree of heterogeneity can be built following the recipe introduced in [49]. The algorithm combines the mechanisms of preferential (with probability α) and uniform random linking ($1 - \alpha$) in such a way that starting from $\alpha = 0$ and increasing its value, the networks generated are successively more homogeneous with a heavy tail whose exponent is equal to ($\alpha = 0$) or larger than ($\alpha > 0$) $\gamma = 3$.

From any initial condition $\{s_i(t=0)\}$, $i = 1, \dots, N$ (where $s_i = 1$ if node i is an instantaneous cooperator and $s_i = 0$ if defector), and after many generations, the instantaneous fraction of cooperators $c(t) = N^{-1} \sum_i s_i(t)$ in the stochastic trajectory, $\{s_i(t)\}$, fluctuates around a well-defined mean value \bar{c} , which depends on the parameter b , as well as on the particular initial condition. The transient time t_0 that we allow before measuring observable quantities is assured to be larger than the one required for the stationarity of \bar{c} (see below). The average level of cooperation $\langle c \rangle$ is computed as the average of \bar{c} over initial conditions (of fixed fraction ρ_0 of cooperators), and network realizations. We numerically identify as pure cooperators all those individuals that *always* cooperate, for all times larger than the transient time t_0 . Pure defectors are those that *always* defect for any $t > t_0$. Fluctuating nodes are those that are neither pure cooperators nor pure defectors. In this way we estimate the measure of the partition sets $(\mathcal{C}, \mathcal{D}, \mathcal{F})$. To inspect the connectedness of the sets of pure strategists, \mathcal{C} and \mathcal{D} , we define cooperator (CC)

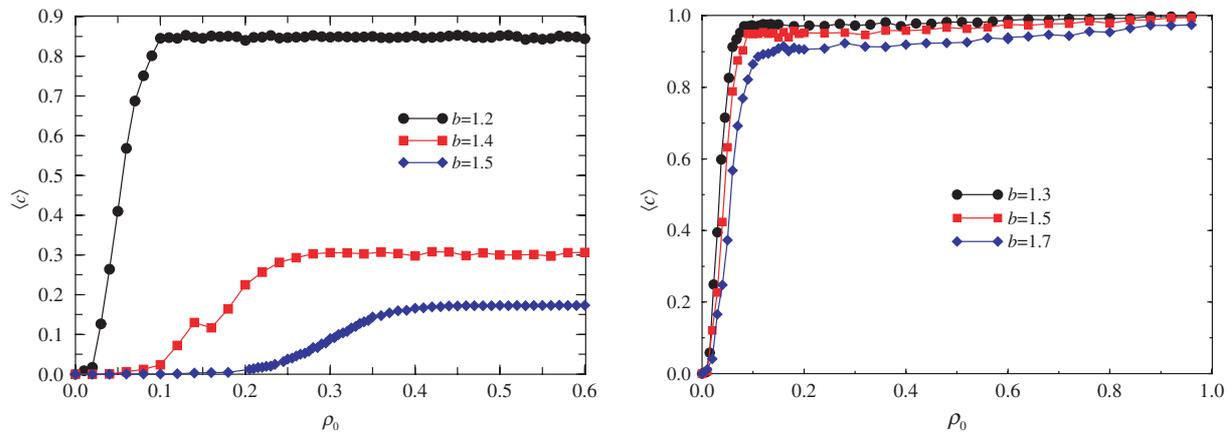


Figure 1. Average cooperation level in ER networks (left panel) and BA networks (right panel) as a function of the initial concentration ρ_0 and several values of b as indicated. The size of the networks is $N = 4000$ nodes and $\langle k \rangle = 8$.

and defector cores (DC) as clusters (connected subgraphs) fully composed by pure cooperators and defectors, respectively, their numbers being denoted by N_{cc} and N_{dc} . It is easy to realize that for generic (irrational) b values, no pure defector can be a neighbour of a pure cooperator, so that the presence of both types of nodes in the long-term stochastic trajectory, assures the existence of fluctuating nodes.

The timescale of microscopic invasion processes (updating rule) is controlled by β^{-1} , which is the highest connectivity of a pair's nodes; this makes that very high payoff of a hub due to its very high k is sensibly balanced by $\beta \propto k^{-1}$ [39], with the side effect that the invasion processes from and to hubs are slowed down, if a hub's (and neighbour's) payoff is much smaller than its connectivity k . On the other hand, the transient time t_0 should be greater than characteristic fixation times for nodes in \mathcal{C} and \mathcal{D} , if one is interested in measuring observable quantities associated to the partition. Fixation times of strategies at the nodes in turn, obviously depend on the initial conditions (i.e. on ρ_0 , the initial fraction of cooperators), so that henceforth in the simulations we use a variable time window, t_0 , of at least 10^4 generations as the transient time. Once the system is at (a fluctuating) equilibrium regarding stationarity of $\langle c \rangle$, we let the dynamics evolve for 10^4 additional time steps, while measuring quantities. All the results have been averaged over at least 10^3 different realizations of the networks and initial conditions. Most of the results shown below correspond to $N = 4000$ nodes, though other values have been also used; we will comment on this issue in the concluding section.

3. Dependence on the initial conditions

The initial conditions for the stochastic trajectories that we consider here are such that an initial number $\rho_0 N$ of nodes ($0 < \rho_0 < 1$) are randomly chosen as cooperators. In figure 1 we show, for some values of the parameter b , how the stationary value of $\langle c \rangle$ depends on the initial fraction ρ_0 of cooperators for ER and BA networks. As seen in that figure, $\langle c \rangle$ typically increases with ρ_0 until saturation is reached much before ρ_0 approaches 1. One observes that saturation occurs sooner for smaller values of b . These features are common for both classes of networks.

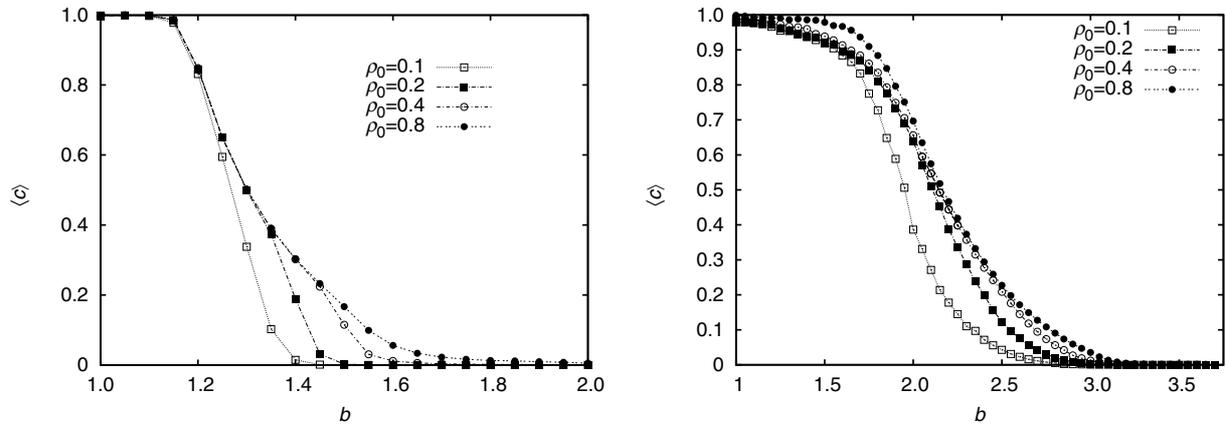


Figure 2. Average cooperation level in ER (left) and SF (right) networks as a function of b and different initial concentration of cooperators ρ_0 as indicated. The size of the networks is $N = 4000$ nodes and $\langle k \rangle = 4$. The SF network is a BA graph whose $P(k) \sim k^{-3}$.

However some details of the $\langle c \rangle(\rho_0)$ curves are different: firstly, for ER network, the departure from zero of $\langle c \rangle(\rho_0)$ occurs, as b increases, only above some (b -dependent) threshold value of the initial fraction of cooperators; on the contrary, for BA networks $\langle c \rangle$ departs from zero as soon as $\rho_0 > 0$, at all values of b inside the coexistence region. Secondly, saturation is more perfect for ER networks, while for BA graphs the plateau in the $\langle c \rangle(\rho_0)$ curve has some small positive slope.

The variation with the game parameter b of the stationary (asymptotic) average cooperation, $\langle c \rangle(b)$, for several values of ρ_0 , is shown in figure 2 for ER graphs (left panel) and BA networks (right panel). In the case of ER networks, different initial concentrations ρ_0 produce a family of curves that mainly differs in their tails in such a way that the larger the value of ρ_0 , the slower the decay of $\langle c \rangle$ as b increases, in correspondence with the perfect saturation of $\langle c \rangle(\rho_0)$ at fixed b observed in figure 1. On the other hand, in BA networks the effects of different initial conditions are appreciated in the whole range of b values. We thus see that degree heterogeneity not only favours the survival of cooperation, but also makes the value of the average cooperation, at fixed b value, more dependent on initial conditions. In this regard one should note that ER networks, often termed as homogeneous, have indeed some small heterogeneity, i.e. the degree distribution density has a nonzero variance. In fact, the average level of cooperation in ER networks is clearly enhanced with respect to random regular networks (where all the nodes have exactly the same degree k), see e.g. [24]. In other words, even the small amounts of heterogeneity of ER networks are enough to allow for cooperation-promoting feedback mechanisms to work.

As stated in the introductory section 1, it has been reported in [41] that for any asymptotic trajectory there is a partition of the network into three sets, namely the set \mathcal{C} of pure cooperator nodes, the set \mathcal{D} of pure defector nodes, and the set \mathcal{F} of fluctuating nodes. From now on we denote by $\rho_c = \langle \mu(\mathcal{C}) \rangle$ the measure (relative size) of the set of pure cooperators (averaged over initial conditions and network realizations), and by $\rho_d = \langle \mu(\mathcal{D}) \rangle$ that of the set of pure defectors. The behaviour of ρ_c and ρ_d versus the game parameter b is plotted in figure 3 for different initial distributions as a function of the parameter b .

The first remarkable result is that in ER networks, the density of pure cooperators does not depend on ρ_0 for the *whole* range of b values, in sharp contrast to the above mentioned

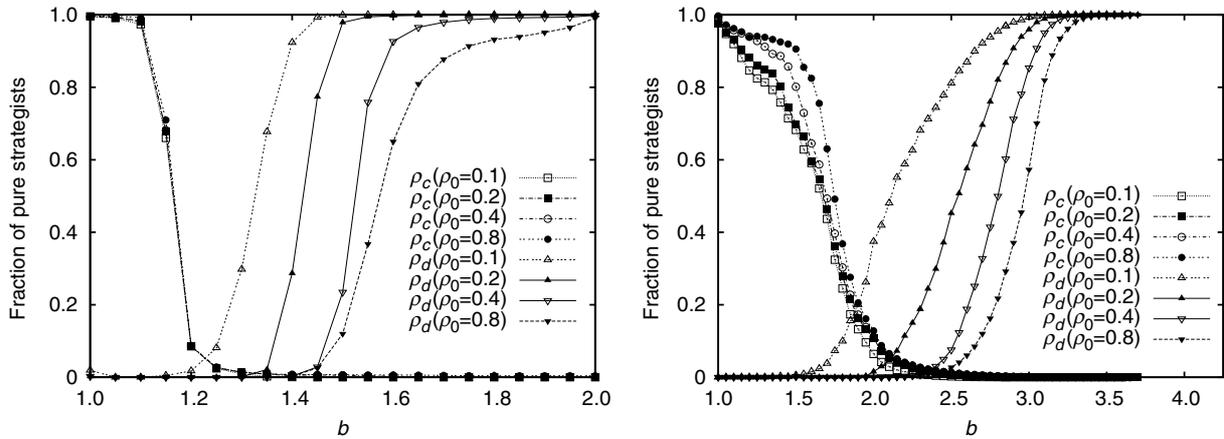


Figure 3. Fraction of pure strategists in ER (left) and SF (right) networks as a function of b and several values of ρ_0 . Network parameters are those of figure 2.

results for the tails of the average level of cooperation $\langle c \rangle(b)$ (see figure 2). As anticipated in the introduction (see equation 1), there are two additive contributions to the average fraction $\langle c \rangle$ of cooperators, namely the measure ρ_c of the set of pure cooperators, and the overall fraction of time \bar{T}_c spent by fluctuating nodes as cooperators, weighted by the relative size $\rho_f = \langle \mu(\mathcal{F}) \rangle$ of the fluctuating set:

$$\langle c \rangle = \rho_c + \rho_f \bar{T}_c. \quad (3)$$

Though the first contribution is, for ER networks, independent of ρ_0 , the second one does indeed depend on initial conditions, as inferred from figure 2 and the relation $\rho_c + \rho_d + \rho_f = 1$. High initial concentrations of cooperators favour the fluctuating set \mathcal{F} at the expense of pure defectors, while the number of nodes where fixation of cooperative strategy occurs remains largely unaffected: ρ_c is thus being mainly determined by the network structural features. For example, in our simulations, for large values of b where ρ_c is very small, we have observed that the pure cooperator nodes form cycles. The fixation of cooperation in these structures is assured if none of their elements is linked to a fluctuating individual that, while playing as a defector, is coupled to more than k_c/b cooperators, where k_c is the number of cooperators attached to the element. The number of such structures is finite in ER graphs, but as soon as their vertices are occupied by cooperators, they will be immune to defectors invasion.

The bottom panel of figure 3 shows the results obtained for BA networks. Regarding the proportion of pure cooperators, one may differentiate two regimes: for $b < 1.7$, there is a moderate dependence of ρ_c on ρ_0 , while ρ_c is almost independent of ρ_0 for larger values of b . This behaviour correlates well with our observations [47] on the distribution of strategists inside the degree classes. In the first range, pure cooperators are present in all k -classes and fluctuating individuals are almost homogeneously disseminated over low-to-intermediate k classes. However, for $b > 1.7$, there is a b -dependent value of k , say k^* , such that k -classes are fully occupied by pure cooperators if $k > k^*$ while basically no pure cooperators are found in lower k -classes. In this second range, where the degree-strategy correlations are strong, the influence of ρ_0 on the asymptotic proportion of pure cooperators is very small.

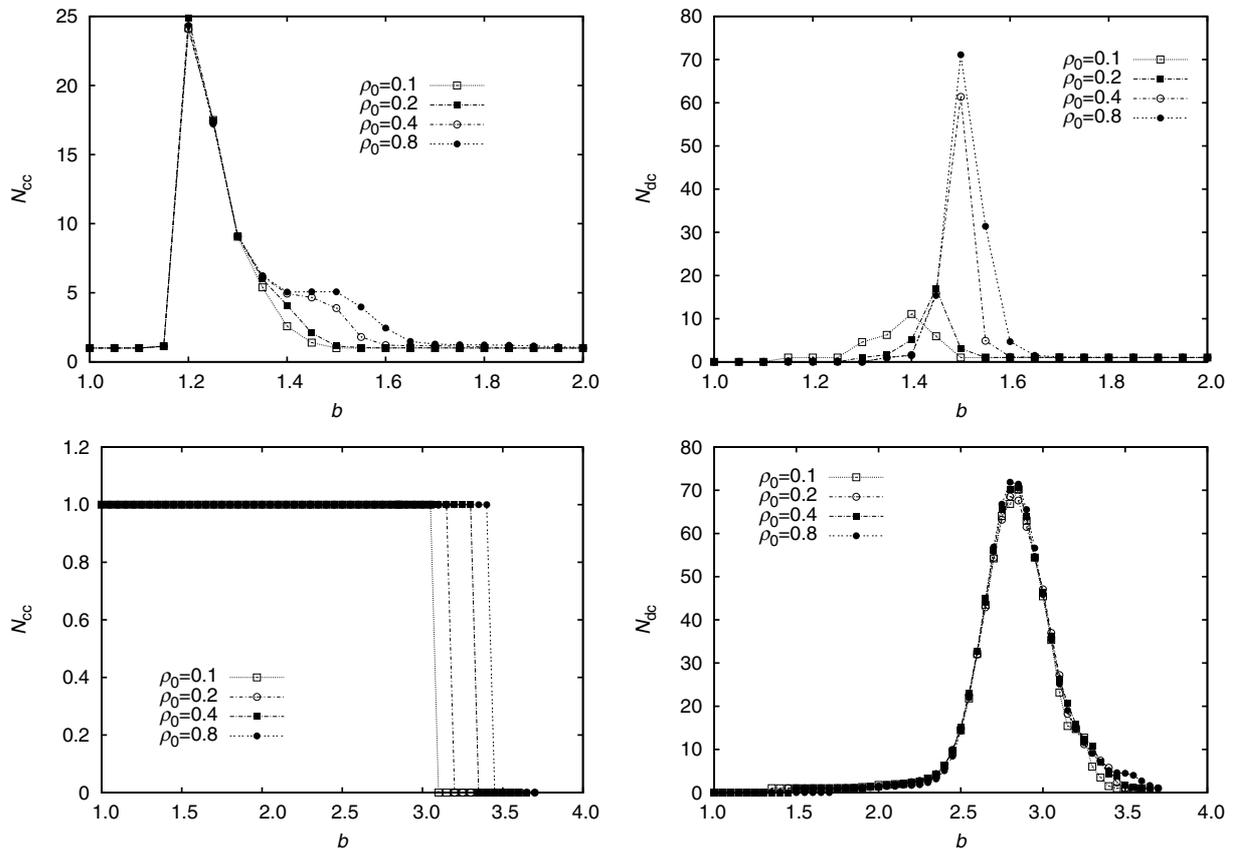


Figure 4. Dependence with b of the number of cooperator (N_{cc}) and defector (N_{dc}) cores in ER graphs (top) and BA (bottom) networks for different values of ρ_0 .

While as discussed in previous paragraphs, the proportion of pure cooperators is either independent (ER) or slightly dependent (BA) on initial concentration ρ_0 , the measures of the other sets in the partition, \mathcal{F} and \mathcal{D} , are indeed more influenced by the initial conditions. The dependence of ρ_d on ρ_0 for BA and ER networks is qualitatively the same, that is, the proportion of pure defectors is favoured (at the expense of the fluctuating set) by a higher initial proportion of defectors. This is consistent with the lack of degree preference (correlation) of pure defectors, which cannot take distinctive advantage of degree inhomogeneity: the higher their instantaneous payoff, the more likely they invade neighbouring nodes, which has the effect of diminishing their future payoff.

Finally, we analyse the connectedness of the pure strategists sets, as measured by the number of cooperator cores N_{cc} , and defector cores N_{dc} . For BA networks, and $\rho_0 = 1/2$, we reported in [41] the result that for all values of b where \mathcal{C} is not an empty set, it is connected, i.e. $N_{cc} = 1$. This result turns out to be independent of ρ_0 (see figure 4). There is only one CC in BA networks, which always contains the most connected nodes, for any initial fraction of cooperators. The grouping of pure cooperators into a single connected set \mathcal{C} allows to keep a significant fraction of pure cooperators isolated from contacts with fluctuating nodes. This ‘Eden of cooperation’ inside \mathcal{C} provides a safe source of benefits to the individuals in the frontier, reinforcing the resilience to

invasion of the set. Pure defectors, on the contrary, do not benefit from grouping together, and the set \mathcal{D} appears fragmented into several DC. Note that for values of $b \simeq 1$, where the set \mathcal{D} is empty, $N_{dc} = 0$, while for very high values of b defection reaches fixation in the whole network, so that $N_{dc} = 1$. Thus, $N_{dc}(b)$ must increase first and then decrease to 1. In figure 4 we show the computed $N_{dc}(b)$ curves for BA networks for several values of ρ_0 . It is remarkable that these curves almost collapse, in spite of the fact that the fraction ρ_d of pure defectors does indeed depend on ρ_0 , a numerical fact for which we have not found a plausible explanation.

In figure 4 we also show for ER graphs $N_{cc}(b)$ and $N_{dc}(b)$, for different values of ρ_0 . Regarding the number of CC, we see that except in the small range $1.4 < b < 1.6$, the different curves coincide, in fair agreement with the independence of ρ_c on initial conditions. Note that in the small interval where they do not coincide, the fraction ρ_c of pure cooperators is below 1%, for all values of ρ_0 . On the other hand, we see that for higher initial proportion ρ_0 of cooperators, the set \mathcal{D} is more fragmented and also that N_{dc} reaches its maximal values at higher values of b .

4. Influence of the degree of heterogeneity

In order to inspect how the results depend on the distribution of nodes' degrees, we have monitored the same magnitudes studied throughout this paper when the value of α varies between 0 and 1. As introduced above, this makes the networks less heterogeneous as α grows and approaches 1. Figure 5 shows, from left to the right, the average level of cooperation $\langle c \rangle$, the density of pure cooperators ρ_c and the density of pure defectors ρ_d as a function of b for several values of α . In this case, the initial distribution of cooperators was set to $\rho_0 = 1/2$, i.e. the nodes have the same probability to cooperate or defect at $t = 0$. The results show that indeed the densities of pure strategists and the average level of cooperation do depend on α , that is to say, the figure confirms the role played by the underlying topology. The more homogeneous the graph is, the smaller the level of cooperation in the system. Moreover, the transition for different values of α is smooth and does not exhibit an abrupt crossover from one kind of behaviour ($\alpha = 0$) to the other ($\alpha = 1$).

We have also explored how nodes where strategies have reached fixation are organized into clusters of cooperation and defection as a function of α . Figure 6 summarizes our computations for the number of CC. In this case, we have represented N_{cc} as a function of $1 - \rho_c$ (that grows with b) in order to have the same scale for different values of α until cooperation breaks down. The observed dependence with α is again smooth and no abrupt change in the behaviour of this magnitude occurs. It is worth stressing that as soon as the underlying network departs from the limit $\alpha = 0$ corresponding to a BA SF network (whose $P(k) \sim k^{-3}$), the number of CC slightly differs from 1. This means that some realizations give rise to more than one cluster of CC. The probability to have such realizations is very small, but in principle, they are possible. As α is further increased beyond zero, it is clear that pure cooperators do not organize anymore into a single CC. We think that this deviation is due to the fact that when $\alpha > 0$ the exponent γ of the underlying network, which still is a SF degree distribution, is larger than 3. It is known that this value of γ marks the frontier of two different behaviours when dynamical processes are run on top of complex heterogeneous networks [50, 51]. This is the case, for instance, for epidemic spreading. For $2 < \gamma \leq 3$, the second moment of the degree distribution $P(k)$ diverges in the thermodynamic limit, while it is finite if $\gamma > 3$. As the critical properties of the system are determined by the ratio between the first (that remains finite for $\gamma > 2$) and the second moment,

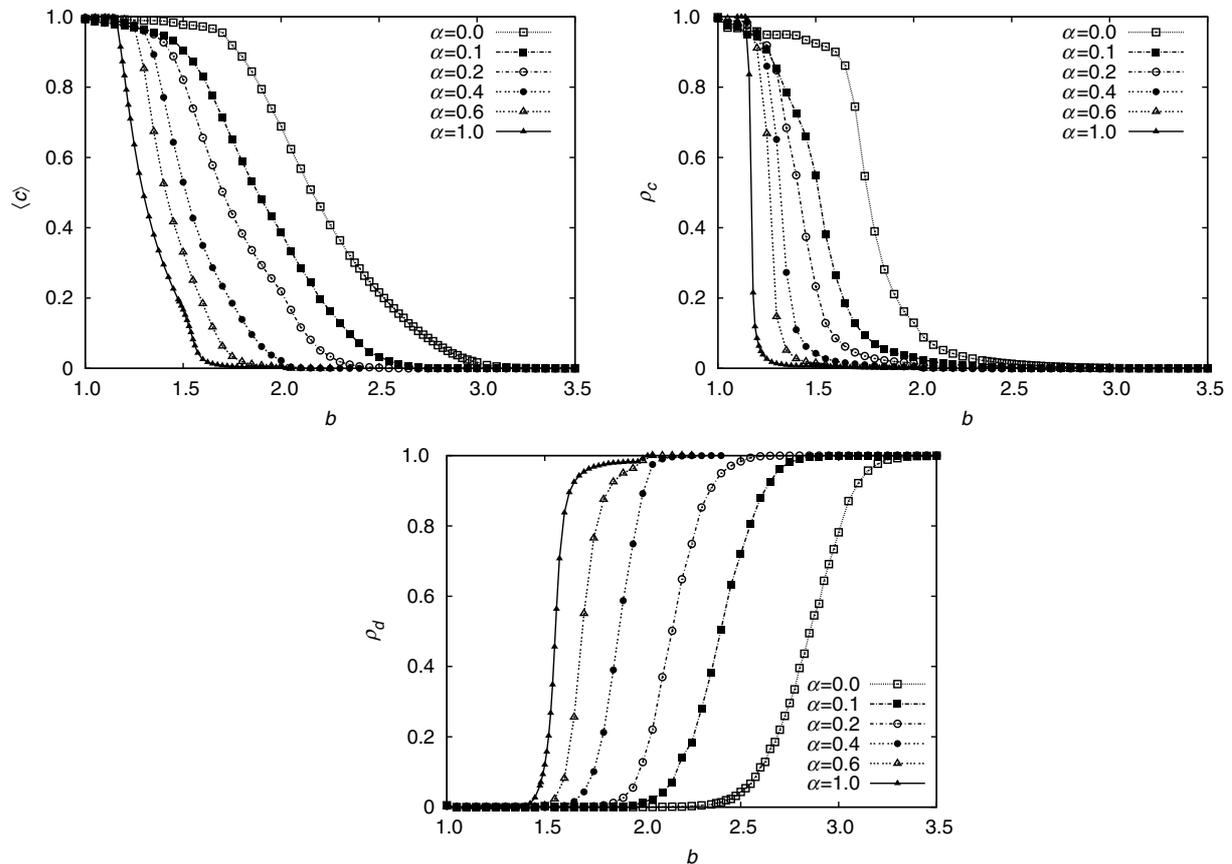


Figure 5. Average cooperation level and densities of strategists as a function of b for different values of α . $\alpha = 0$ corresponds to a BA network while $\alpha = 1$ generates an ER graph. In this case, the networks are made up of $N = 2000$ nodes and $\langle k \rangle = 4$. See the main text for further details.

the divergence of the latter when $N \rightarrow \infty$ and $2 < \gamma \leq 3$, makes the epidemic threshold null. On the contrary, when the process takes place in networks whose $\gamma > 3$, the epidemic threshold is recovered, although no singular behaviour is associated with the critical point [50, 51]. We expect that a similar phenomenology is behind the results shown in figure 6. It would be very interesting to test this hypothesis by simulating the PD implemented here on top of SF networks with an exponent in between 2 and 3. As a by-product, such a study may guide our search for analytical insights and provide a deeper understanding of what drives the structural organization of cooperation at the microscopic level.

5. Conclusions

SF-structured populations offer to the cooperative strategy the opportunity of positive feedback evolutionary mechanisms making cooperation the most fit overall strategy, in spite of not being the best reply to itself in one-time step. We have shown here that the enhancement of cooperation due to the heterogeneity of the structure of connections among agents is robust against variation

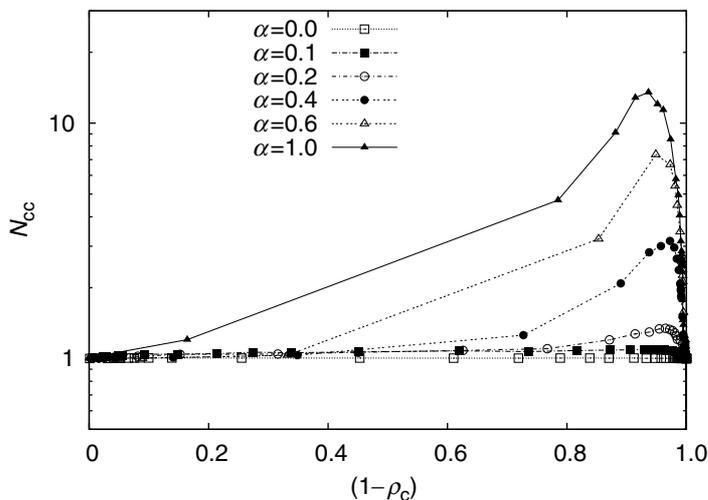


Figure 6. CC for different networks defined by the value of α as a function of the density of nodes that are not pure cooperators $1 - \rho_c$. Network parameters are those used in figure 5.

of initial conditions (initial concentration ρ_0 of cooperators): while both the measure of the set \mathcal{C} where cooperation reaches fixation, and its connectedness properties are either independent or only slightly dependent on ρ_0 , the measure of the fluctuating set \mathcal{F} and the set \mathcal{D} where defection is fixed, both show a clear dependence on initial conditions, for defection cannot profit from degree heterogeneity. On the other hand, the characteristics of the asymptotic evolutionary states of the PD analysed here show a smooth variation when the heterogeneity of the network of interconnections is one-parametric tuned from Poissonian to SF, demonstrating a strong correlation between heterogeneity and cooperation enhancement.

Though the numerical results presented here correspond to network sizes $N = 4000$ (in section 3) and $N = 2000$ (section 4), we have also studied larger networks (up to $N = 10^4$), with no qualitative differences in the results. The increase of network size, while keeping the average degree $\langle k \rangle$ constant, turns out to be beneficial for cooperation, due to the fact that it has the effect of increasing the maximal degree, and thus the range of degree values. This further confirms how efficiently cooperation takes advantage from degree heterogeneity.

The robustness of these results against game parameters variation will be analysed elsewhere [47], one should expect that the network partition (\mathcal{C} , \mathcal{D} , \mathcal{F}) along asymptotic stochastic trajectories is generic in evolutionary game dynamics in graphs, for the kind of stochastic updating rule considered here. Our results also suggest that more works are needed in order to fully characterize the behaviour of the PD game in heterogeneous graphs. The use of real networks, with emphasis on the role of mesoscopic (community) structures is addressed in [46]. Of particular interest would be to perform the sort of analysis carried out here in SF networks with an exponent $2 < \gamma < 3$, which will make it feasible to connect evolutionary dynamics with other dynamical processes taking place on top of SF networks. Our hope is that this sort of study might provide a deeper understanding of what is going on at the microscopic level and might help to comprehend what universal mechanisms drive the evolution of complex heterogeneous networks as well as the reasons behind their ubiquitous presence in nature.

Acknowledgments

We thank A Arenas, J M Pacheco and A Sánchez for helpful comments and discussions. JGG and YM are supported by MEC through a FPU grant and the Ramón y Cajal Program, respectively. This work has been partially supported by the Spanish DGICYT Projects FIS2006-12781-C02-01 and FIS2005-00337.

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