



Self-organized criticality in a fibre-bundle-type model

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Abstract

The dynamics of a fibre-bundle-type model with equal load sharing rule is numerically studied. The system, formed by N elements, is driven by a slow increase of the load upon it which is removed in a novel way through internal transfers to the elements broken during avalanches. When an avalanche ends, failed elements are regenerated with strengths taken from a probability distribution. For a large N and certain restrictions on the distribution of individual strengths, the system reaches a self-organized critical state where the spectrum of avalanche sizes is a power law with an exponent $\tau \simeq 1.5$. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Twelve years ago the idea of self-organized criticality (SOC) was introduced by Bak et al. [1], as a way of understanding the fractal structure and the $1/f$ noise behaviour displayed by a wide variety of large interactive systems. Although a precise definition of SOC is still lacking, many papers on the subject have appeared [2] and avalanche-like behaviour has been experimentally observed in many real physical phenomena: microfracturing processes [3], earthquakes [4], fluid flow through porous media [5], flux lines in superconductors [6], etc. Among the number of proposed models to describe SOC behaviour, the sandpile [1], forest-fire [7,8], invasion percolation [9], Bak-Sneppen [10] and Olami, Feder and Christensen (OFC) [11] type models have been perhaps the most intensively studied, constituting paradigms in this subject.

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In the present paper we make use of the well-known fibre-bundle models widely used to analyze the fracture process in heterogeneous materials [12–14]. They have also been applied in geophysics [15,16]. In fibre-bundle models, a set of elements is located on a supporting lattice, each with a strength threshold sampled from a probability distribution. In these models, once an element fails, its load is distributed among the surviving elements. Different load transfer rules can be defined depending on the range of the interaction assumed. In the ELS (for equal load sharing) case, the load carried by a failed element is equally distributed among the surviving elements of the system, representing in this way a long-range interaction. The ELS model is a sort of mean field approximation to the more realistic local transfer schemes. It has been found that the distribution of avalanche sizes in a breaking cycle in the static version of the ELS model follows a power law [17,18]. It is clear that this is not at all a model of the SOC type, because in the breaking process a stationary state cannot be reached as the broken elements remain broken during the cycle. In what follows we propose a fibre-bundle model which does exhibit SOC behaviour by using an ELS transfer rule, a novel way for dissipation, and the hypothesis that the failed elements after an avalanche are regenerated, i.e., they are assigned new strength thresholds. In Section 2 we present the model. Section 3 is devoted to the results obtained from simulations and to discussion. Our conclusions are given in Section 4.

2. The model

Let there be a set of N elements located on a supporting lattice. Suppose that each element carries a given load σ and has a strength threshold σ_{th} . This can be viewed as a representation of a disordered material in which each small volume is described by its breaking characteristics. In order to assign the random thresholds, different probability distributions can be considered. In materials science the Weibull distribution is usually used,

$$P(\sigma) = 1 - e^{-(\sigma/\sigma_0)^\rho},$$

where ρ , being the so-called Weibull index, controls the degree of threshold disorder in the system (the bigger the Weibull index, the narrower the range of threshold values), and σ_0 is a load of reference which acts as unity. In the following, we will assume $\sigma_0 = 1$, and therefore the loads and thresholds used henceforth are dimensionless. Thus, to each site i , $1 \leq i \leq N$, one assigns a random threshold value $\sigma_{i_{\text{th}}}$:

$$n_i = 1 - e^{-\sigma_{i_{\text{th}}}^\rho}, \quad (1)$$

where n_i are random numbers uniformly distributed between 0 and 1.

At the beginning, the load carried by all the elements is set to zero. In analogy with the OFC model [11] the system is driven at the same rate. During each *external* time step of loading, all the elements in the set increase their load by a small amount v ,

$$\sigma_i \rightarrow \sigma_i + v, \quad \forall i. \quad (2)$$

This mode of driving the system allows us to obtain the limit of infinitesimal driving rate. In practice, we search for the smallest threshold value and add this amount to all the elements of the system. This makes, at least, one element critical. Suppose that as a consequence of applying (2), q_1 elements become unstable (usually, $q_1 = 1$). The homogeneous drive is switched off, the unstable elements fail and the following relaxing rule is applied to all the q_1 elements:

$$\sigma_i \rightarrow 0, \quad \forall \sigma_i \geq \sigma_{\text{th}}.$$

Now, assuming an ELS transfer rule, the total load supported by the q_1 elements, $\sigma_{\text{dist}} = \sum_{i=1}^{q_1} \sigma_i$, is equally distributed among all the remaining elements (the $N - q_1$ surviving elements), so that the new load on all the surviving elements is

$$\sigma_i \rightarrow \sigma_i + \frac{\sigma_{\text{dist}}}{N - q_1}, \quad \forall \sigma_i < \sigma_{\text{th}}.$$

This may have the effect that other elements become unstable and the avalanche continues. This case will be commented on in the next paragraph. If this is not the case, the broken elements are regenerated with new random threshold values and zero load, and rule (2) is repeated until a new avalanche is triggered. We define the size, s , of an avalanche as the number of broken elements between two successive steps of external loading of the system, an *internal* time step as a visit to all the N elements of the set checking whether or not their σ -value is larger than or equal to their σ_{th} -value, and the avalanche lifetime, T , as the number of internal time steps needed for the system to be completely relaxed.

Now let us assume that as a consequence of the distribution of the amount σ_{dist} , q_2 elements became overcritical, i.e., $\sigma_i \geq \sigma_{\text{th}}$, for these q_2 elements. Being part of the same avalanche, the q_1 elements broken before are not regenerated and the new amount of load to be distributed is $\sigma_{\text{dist}} = \sum_{i=1}^{q_2} \sigma_i$. As mentioned before, the ELS transfer scheme implies that the load supported by failing elements is equally distributed among the surviving elements of the set. On the other hand, it is clear that to make possible the existence of a stationary state one has to introduce an exit of load to make it possible that, on an average, the load inflow is compensated by an outflow from the system. We will assume that the system loses load through the elements that have previously failed in the *same* avalanche, that is, when the transfer of the load carried by currently failing elements takes place, the portion of load that corresponds to the already broken elements in previous internal time steps of the same avalanche leaves the system. This is a novel way for dissipation and plays the role of the boundaries in other models of SOC. Its physical meaning is straightforward: regions that have just failed in that avalanche cannot accumulate stress during the same fracture process. This assumption implies that the total amount of load removed from the system after an avalanche has ended, depends on both the avalanche size and on its lifetime.

Continuing the process, the amount σ_{dist} is distributed among all the $N - q_2$ elements which remain as spectators in this second internal time step of the breaking process, that is, the q_1 elements broken in the first internal time step, and the remaining $N - q_2 - q_1$

elements which are stable. Hence, the update for the surviving elements is performed according to:

$$\sigma_i \rightarrow \sigma_i + \frac{\sigma_{\text{dist}}}{N - q_2}, \quad \forall \sigma_i < \sigma_{\text{th}} \quad (3)$$

and the load $\sigma_{\text{lost}} = q_1 \sigma_{\text{dist}} / (N - q_2)$, corresponding to the q_1 broken elements failed in the first internal time step, is lost. The surviving elements are checked again. If, for example, q_3 new elements become unstable in this third internal time step, $[1/(N - q_3)] \sum_i^{q_3} \sigma_i$ units of load are added to the remaining $N - q_3 - q_2 - q_1$ surviving elements and $\sigma_{\text{lost}} = [(q_2 + q_1)/(N - q_3)] \sum_{i=1}^{q_3} \sigma_i$ units of load are lost in this third internal time step. We check if new elements become unstable and so on. The process continues until we regain a static state ($\sigma_i < \sigma_{\text{th}}$ for all the surviving elements) where the avalanche ends. The broken elements are regenerated with new randomly chosen strength values and with loads equal to zero. In the example given above, if no elements become unstable when the third distribution of load takes place, the avalanche stops and its size and lifetime are $s = q_1 + q_2 + q_3$ and $T = 3$, respectively.

3. Results and discussion

Two distinct behaviours of the system explained in Section 2 are obtained from numerical simulations according to the width of the probability distribution from which the strength thresholds are taken. In the first, the system does not exhibit the characteristics of SOC behaviour although the avalanche size distribution for small avalanches is a power law, while in the second one the system is able to settle into a stationary state with fluctuations around the temporal mean value of the system load, and with power-law distributions for both the avalanche sizes (over the entire range of avalanche sizes) and the avalanche lifetimes.

3.1. Non-SOC behaviour

For large values of ρ , the Weibull distribution is sharply peaked. Thus, irrespective of the system size, there are many elements in the set with similar breaking properties, i.e., with very close strength threshold values. With the hypotheses of our model, it provokes a simple pattern of dynamical evolution: periods of slow loading followed by catastrophic avalanches. This is observed in Fig. 1, where we have plotted the value of the mean load per element stored in the system as a function of the number of avalanches for a system of $N = 10\,000$ elements and $\rho = 4$. As can be observed, the average value of the system load does not reach a statistically stationary value but systematically increases with time and suddenly falls to zero, with a sort of quasi-periodic sawtooth behaviour (the pattern is not completely periodic, because there is some fluctuation in the amplitude of the drop and in the time interval between major avalanches).

We have also monitored the distribution of avalanche sizes. The results obtained are similar to those reported in Ref. [19], where the dynamics of a sandpile-like model was

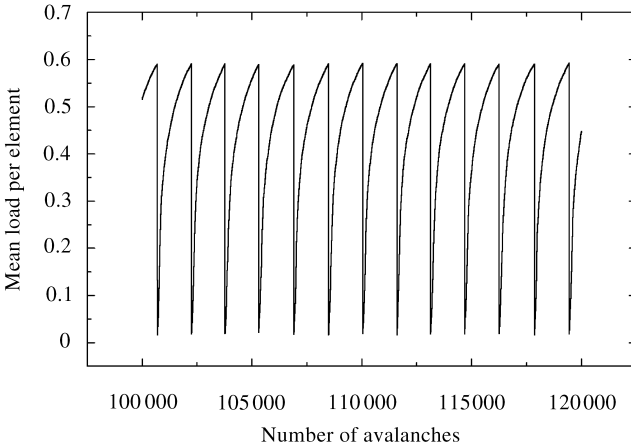


Fig. 1. Dimensionless mean load per element stored in the surviving elements after an avalanche ends, as a function of the number of avalanches ($\rho = 4$). Note the quasi-periodic sawtooth behaviour. This is a typical graph for the non-SOC state.

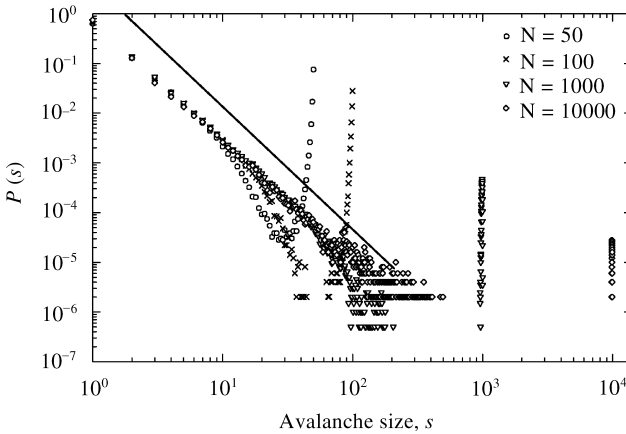


Fig. 2. Avalanche size distributions for different system sizes ($\rho = 4$). The solid line has a slope of -2.5 .

investigated, and with those of Refs. [17,18]. Fig. 2 shows, in dimensionless units, the avalanche size distribution for system sizes of $N=50, 100, 1000$ and $10\,000$ with $\rho=4$. Two distinct features are observed. For the smallest system size there are avalanches of almost all sizes and the distribution has a sharp peak for large values of avalanche sizes. Increasing the system size produces a gap in the avalanche spectrum. Now, the event size distribution is bimodal: at the small scale, the spectrum of small avalanches is of the power law type over a reduced range of avalanche sizes, whereas at the largest scale there is an excess of events whose sizes are of the order of the system size.

The absence of avalanches for intermediate sizes is clear and can be interpreted as being due to the simultaneous failure of many elements with very close threshold values

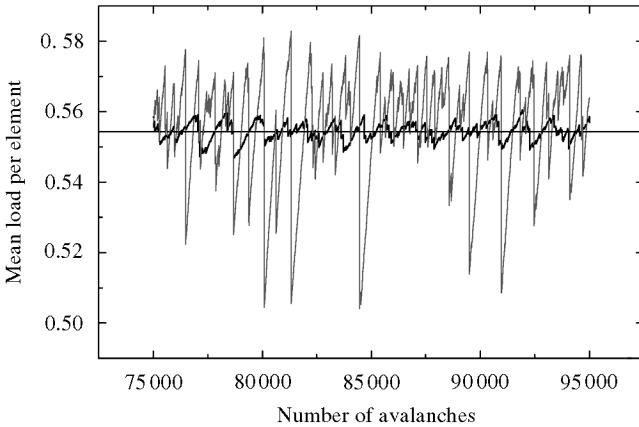


Fig. 3. Mean load per element carried by surviving elements when an avalanche ends as a function of the number of avalanches. Fluctuations around an average value (represented by the horizontal solid line) decrease as the size of the system increases from $N = 10\,000$ elements (dotted line) to $N = 50\,000$ elements (solid line). $\rho = 2$ in both cases.

whose failure triggers catastrophic events of large sizes. It is interesting to note that whenever a power law distribution can be identified, its exponent is of about $\frac{5}{2}$, i.e., the same reported from an analytic study for the burst distribution in static fibre-bundle models with ELS transfer rule [17,18]. This is understandable because the behaviour of our model in this non-SOC regime is similar to a succession of breakings of static ELS fibre-bundle sets. At the beginning, small avalanches produced are randomly dispersed throughout the system; then a crack is nucleated that leads to a final, catastrophic avalanche where an important fraction of N fails at the same instant. As our system dissipates through the elements broken during the ongoing avalanche, this final catastrophic avalanche unloads the system very efficiently, leading it to the beginning of a new cycle of slow loading. This process (Fig. 1) could be called the “cistern effect”, because of its similarity with the familiar rate of filling and flushing of an old-fashioned toilet cistern.

3.2. SOC behaviour

For ρ values such that the width of the Weibull distribution is wider, the system, without any tuning, is able to self-organize into a stationary state where the flow of load into the system equals the flow of load out of the system. This is due to the large inhomogenities in the distribution of threshold values, that is, the existence of both highly resistant elements together with other very brittle elements. Fig. 3 shows the evolution of the load per element accumulated by the whole system as a function of the number of avalanches for systems of $N = 10\,000$ and $50\,000$ elements with $\rho = 2$ in both cases. As can be seen, the average value of the stored load fluctuates around a temporal mean value and these fluctuations decrease as the size of the system increases. This stationary state is reached if the size of the system is large enough. For

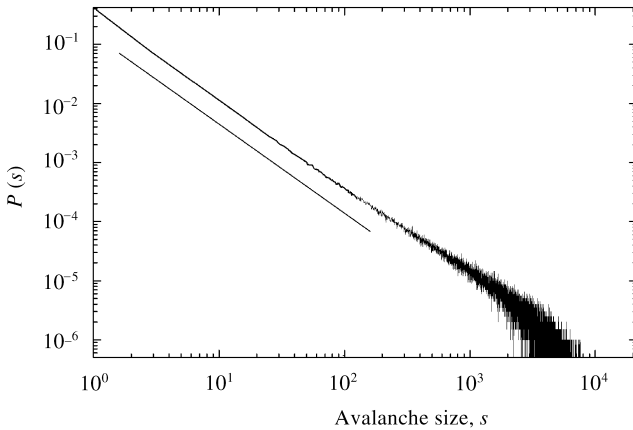


Fig. 4. Typical graph of the avalanche size distribution in the SOC regime. This case is for a system of $N = 50\,000$ elements and $\rho = 2$. The straight line has a slope $-\tau = -1.5$.

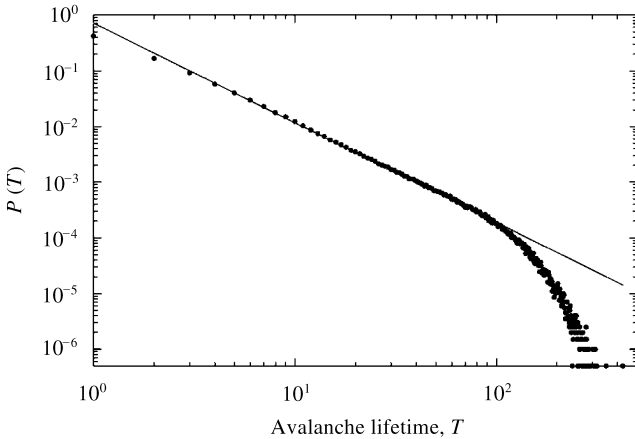


Fig. 5. Avalanche lifetime distribution for a system consisting of $N = 50\,000$ elements and $\rho = 2$. The slope of the solid line is $-y = -1.81$.

a too small N there exists a background of avalanches, with sizes of the order of the system size, which frequently provokes total collapses. This fact agrees with the basic assumption that SOC behaviour demands large systems.

In Fig. 4 we have plotted the avalanche size distribution for a system with $N = 50\,000$ elements and $\rho = 2$. A power law of the form $P(s) \sim s^{-\tau}$ can be very well fitted over the entire range of event sizes with a critical exponent τ close to 1.5. We have also found a power-law $P(T) \sim T^{-y}$ for the distribution of avalanche lifetimes T , shown in Fig. 5 for values as in Fig. 4. Table 1 summarizes the results obtained for different values of ρ in a system of $N = 50\,000$ elements.

We have checked that once the system fulfills the SOC behaviour (appropriate ρ , and large enough N), the critical exponents obtained for both the avalanche size distribution

Table 1
Critical exponents for systems of
 $N = 50\,000$ elements and different
Weibull index ρ

ρ	τ	y
1	1.57	1.96
1.5	1.55	1.88
2	1.55	1.81
2.1	1.55	1.89

Table 2
Critical exponents for systems of $N = 50\,000$ elements and different values of the Power-law index m

m	τ	y
2	1.59	2.06
2.5	1.61	2.01
3	1.60	1.95
3.5	1.59	1.97

and the avalanche lifetime distribution do not vary with the system size. The exponent y characterizing the avalanche lifetime distribution slightly varies with the Weibull index. However, the value of the critical exponent τ for the distribution of avalanche sizes is universal, that is, it does not depend on the ρ value. The result $\tau \sim 1.5$ is close to the value derived from mean field approximations for SOC systems [20,21]. This fact is not surprising because as mentioned above the ELS transfer scheme is itself a sort of mean field-like approximation.

We have performed numerical simulations of this model, changing the Weibull probability distribution by a power-law (Pareto-like) distribution,

$$P(\sigma) = 1 - \frac{1}{\left(\frac{\sigma}{\sigma_0}\right)^m + 1},$$

where m and σ_0 are the parameters of the distribution. In this case m plays the role of ρ in the Weibull distribution and σ_0 is again a load of reference which is set to one in the numerical simulations. Our results are shown in Table 2. The dynamical evolution of the system is qualitatively similar and the avalanche size distribution shows a power law with, again, a critical exponent $\tau \simeq 1.5$.

4. Conclusions

We have introduced a fibre-bundle-type model with an ELS transfer rule and a new way for load removal from the system. The qualitative results obtained for the dynamics of the model are summarized in a type of two-phase diagram as can be seen in Fig. 6.

For small values of N , no matter how the threshold values are distributed, the dynamics of the system is of the type of the static fibre-bundle model with ELS

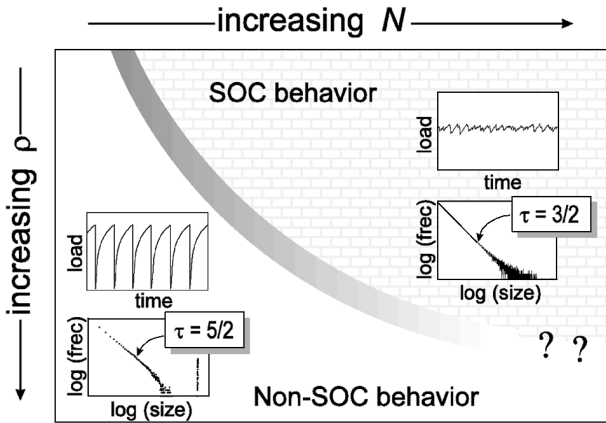


Fig. 6. Schematic phase diagram of the system. Inset graphs summarize typical avalanche size distribution and load fluctuations for the two regimes.

transfer rule. Increasing the value of ρ , even for large N the system is unable to avoid a quasiperiodic sequence of complete failure. This non-SOC zone is characterized by an avalanche size distribution with two clear features: one for the smaller scales in which the avalanche size distribution is of the power-law type with a critical exponent $\tau = 2.5$ and the other for the bigger scales with a sharply peaked distribution in the neighbourhood of N .

On the other hand, for large enough N and in the range of moderate ρ -values corresponding to large inhomogeneities in the threshold values of the elements, the system self-organizes into a statistically stationary state characterized by power law distributions for the size and duration of the avalanches. We have found that the critical exponent characterizing the avalanche size distribution is $\tau \sim 1.5$, very close to the value derived from mean field arguments for SOC systems, and that it is universal, that is, it does not depend on the probability distribution from which the threshold values are taken. Finally, we shall say that for large values of ρ and N the exploration of the dynamics of the system becomes very difficult because of the very long transient period needed before reliable conclusions can be obtained.

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