

Probabilistic approach to time-dependent load-transfer models of fracture

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A probabilistic method for solving time-dependent load-transfer models of fracture is developed. It is applicable to any rule of load redistribution, i.e., local, hierarchical, etc. In the new method, the fluctuations are generated during the breaking process (annealed randomness) while in the usual method, the random lifetimes are fixed at the beginning (quenched disorder). Both approaches are equivalent. [S1063-651X(98)09708-6]

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I. INTRODUCTION

The modelization of fracture in disordered systems is a subject of great interest in natural and artificial materials [1]. A time-dependent method to describe the failure of materials under stress, within the fiber-bundle paradigm, was proposed by Coleman [2]. In this model a set (bundle) of elements (fibers) is considered with each element having a prescribed lifetime when subject to an applied stress (load). When elements fail, their load is redistributed to other elements of the set according to a prescribed rule of transfer. As a consequence of the load transfer, the lifetime of the receptors is actually reduced and the question is: How long does it take for the whole set to collapse? These fiber-bundle models are called dynamical, or time dependent [2–4], as opposed to their static counterparts, which have also been intensively studied [5–7]. The rule for redistributing the load of failed elements can be wide, but there are two limiting cases. In the first, the stress of the failed element is transferred equally to all surviving elements (ELS, for equal load sharing). In the second, the load of the failed element is transferred to the nearest surviving element(s) (LLS, for local load sharing). Hierarchically organized transfer (HLS) criteria are also of great interest [7–9]. Recently, these models have received much attention in the geophysical literature [10], because one would reasonably expect the emergence of universal scaling laws of the type observed in seismology [11,12]. In this field, the bundle is a representation of a fault, and the individual elements or fibers represent asperities on the fault plane.

In Ref. [12], the ELS case is formulated in terms of a differential equation of the radioactive decay type. We have followed this perspective to devise a numerical probabilistic method to deal with any type of transfer rule. In Sec. II, we explain in detail the differences between the usual approach and the new probabilistic approach. In Sec. III, we compare results and present a brief discussion.

II. THE METHODS

Suppose a set of N_0 elements identified on the sites of a supporting lattice. This information is contained in a list

$\{\vec{x}_{i,t}\} \equiv \{\vec{x}\}_t$, $1 \leq i \leq N_0$. This list is necessary, except in the ELS case, in order to know how to distribute the load of the failed elements. The broken elements are marked by $\{\vec{x}\}_t$. At $t=0$, all the elements of the set are loaded with a reference value $\sigma_0 = 1$. At any time, the total load acting on the surviving elements is constant, equal to $N_0\sigma_0$.

To each element i one assigns a random lifetime, $t_{i,0}$, under the unity of stress:

$$n_i = 1 - e^{-t_{i,0}}, \quad (2.1)$$

where n_i are random numbers between 0 and 1. This choice implies a logarithmic distribution of lifetimes. A more general distribution function for the failure time of a single element subjected to a known load history $\sigma(t)$ is (see, for example [4,11])

$$n_i = 1 - \exp \left[-\Psi \left(\int_0^{t_{i,0}} \nu[\sigma(\tau)] d\tau \right) \right], \quad (2.2)$$

where $\Psi(x)$ is the *shape function*. The time integral in Eq. (2.2) introduces a hazard rate $\nu(\sigma)$ known generally as the *breakdown rule* in terms of the instantaneous load level. Experimental and theoretical work [2,4] favors a shape function $\Psi(x)$ of the form

$$\Psi(x) = x^\beta, \quad (2.3)$$

known as the Weibull shape function, with the particular choice $\beta=1$ giving the exponential shape function.

As for the breakdown rule, two special forms are widely used in the literature: the exponential breakdown rule,

$$\nu_e = \phi e^{\eta(\sigma/\sigma_0)}, \quad (2.4)$$

and the power-law breakdown rule,

$$\nu_p = \nu_0 \left(\frac{\sigma}{\sigma_0} \right)^\rho, \quad (2.5)$$

with ϕ , η , ρ , ν_0 , σ_0 all positive constants. ρ is called the Weibull exponent because inserting Eq. (2.3) and Eq. (2.5) in Eq. (2.2) mimics the static Weibull distribution for the failure load of a single element. This parameter typically varies between 2 and 5. The exponential breakdown rule, Eq. (2.4),

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has a characteristic failure rate, whereas the power-law breakdown rule is scale free and can be regarded as a local approximation to the former. Following [12], we will use Eq. (2.5) as the individual breakdown rule in order to be able to compare the performance of the two approaches for load-transfer models. For further insights into the theoretical and experimental basis of the Weibull shape function see [2], and for the breakdown rules see Refs. [4] and [11].

Without losing generality one can choose $\nu_0 = \sigma_0 = 1$ in Eq. (2.5), which means that σ^ρ is a measure of the failure rate (i.e., a unit failure rate under the unity of load is assumed). As ν_0 is actually a frequency, $\nu_0 t$ is a dimensionless time variable and because of the particular choice $\nu_0 = 1$, t will hereafter stand for nondimensional time.

If one substitutes Eq. (2.3) and Eq. (2.5) in Eq. (2.2) with the particular set of constants $\beta = \nu_0 = \sigma_0 = 1$, we obtain

$$n_i = 1 - \exp\left(-\int_0^{t_{i,0}} \sigma^\rho(\tau) d\tau\right), \quad (2.6)$$

which can be integrated for constant unit load $\sigma(t) = \sigma_0 = 1$ to give Eq. (2.1).

When loads of failed elements are redistributed, the load acting on each element will no longer be the constant σ_0 but will depend on time $\sigma_i(t) \geq \sigma_0 = 1$. Thus we introduce a reduced time to failure for each element, $T_{i,f}$, given by

$$t_{i,0} = \int_0^{T_{i,f}} \left[\frac{\sigma_i(t)}{\sigma_0}\right]^\rho dt. \quad (2.7)$$

In the case of independent elements, $\sigma_i(t) = \sigma_0 = 1$ and $t_{i,0} = T_{i,f}$. However, load transfer occurs, and hence the actual time to failure of element i , $T_{i,f}$ is reduced to below $t_{i,0}$. By imposing the fulfillment of Eq. (2.7), the successive order of breaking of the N_0 elements, one after the other, is easily identified and the total time of collapse is the $T_{i,f}$ of the longest lasting element. Thus, in this approach the randomness, that is the population of lifetimes, is fixed at $t=0$ (quenched disorder), and the breaking process is completely deterministic. Henceforth, we will refer to this approach as the usual one.

In the new probabilistic approach presented here, the fluctuations are generated during the breaking process and hence it is an example of so-called annealed randomness. An interesting question is whether the two types of disorder, namely, quenched and annealed disorder, in these models lead to different results, as has been observed for some critical phenomena [1].

In Ref. [12], Newman *et al.* formulated the ELS mode in terms of a differential equation of the radioactive decay type. Denoting the number of surviving elements as $N_s(t)$, its differential change is given by

$$\frac{dN_s}{dt} = -N_s \sigma^\rho, \quad (2.8)$$

hence σ^ρ represents the decay rate. But in the ELS mode $\sigma = (N_0/N_s)$, hence

$$N_s(t) = N_0[\rho(T_f - t)]^{1/\rho} \quad (2.9)$$

and $T_f = 1/\rho$. In this setting, fluctuations do not exist and one simply obtains mean values for the failure rate.

Following the perspective of the previous differential equation in which a group of elements, supporting the same load σ , fail at a rate σ^ρ or, in other words, have a mean life $1/\sigma^\rho$, one can devise a probabilistic method for any transfer rule. The scenario would be a set of N_0 elements identified in $\{\vec{x}\}_t$, like a sample of radioactive nuclei fixed on a lattice, all having initially a decay rate $\sigma_0^\rho = 1$. As time passes, failures (disintegrations) occur and this does not merely imply the effective disappearance of the failed elements, but also the modification of the decay rate of other surviving elements. The modification comes from the redistribution of load as accorded in the rule of transfer (ELS, LLS, etc.), and the assumption that the decay rate of any element is given by its σ^ρ value. As in this strategy of calculation one has to proceed at discrete time intervals, δ_j , $j=1,2,\dots$, the information of loads in the set will be contained in a list denoted by $\{\sigma_{i,j}\} \equiv \{\sigma\}_j$. The list is updated at each time step, together with $\{\vec{x}\}_j$. After j time steps, there will have appeared subsets of elements. Each subset is formed by all the surviving elements bearing the same load. We organize these subsets into sublists identified by the subindex l , and denote the corresponding load by Y_l and the number of elements belonging to the sublist l by N_l . This information, which is obtained from $\{\sigma\}_j$, will be denoted by $\{Y_l, N_l\}_j$ and updated simultaneously. At the beginning, as the load of all elements is 1, the sublists are

$$Y_{l,0} = \begin{cases} 1, & N_{l,0} = N_0 & \text{if } l=1, \\ 0, & N_{l,0} = 0 & \text{if } l \neq 1. \end{cases}$$

Now, it is clear that the simultaneous existence of several sublists in the sample, each with a different decay rate Y_l^ρ , poses a difficulty for an accurate description of the decay process of the whole set [13]. The key point is the choice of the length of the time intervals, δ_j . To illustrate this problem, in Fig. 1 we have plotted the detailed evolution of breaking of a hierarchical set of $N_0 = 1024$, coordination number equal to 2, and $\rho = 4$. In abscissas one represents time, from 0 to T_f . In ordinates the spatial position of the N_0 elements of the set is represented. At $t=0$ all elements are sound. As time evolves, breakings (represented by small crosses) are produced and therefore the accumulated number of failed elements, represented by the continuous line, grows. At $t=T_f$ the number of failures is N_0 . The height of the vertical spikes represents the load supported by an element at the time of failing. For short times, ruptures appear dispersed across the set and the rate of breaking is small. Progressively though, there appear cracks formed by the failures of neighboring elements, and this makes the continuous line adopt steeper slopes. Finally, the final breakdown occurs related to a big crack of a size similar to the whole system. This stage is also related to the high values of the spikes. The progressive acceleration of the breaking process is thus clear from this figure.

Therefore the time interval used in the probabilistic approach must be variable with time. Otherwise, if one takes for δ a reasonable value for the beginning, the final part of the breaking will be badly described: in each δ , many elements will fail and the prediction of T_f will be very inaccurate.

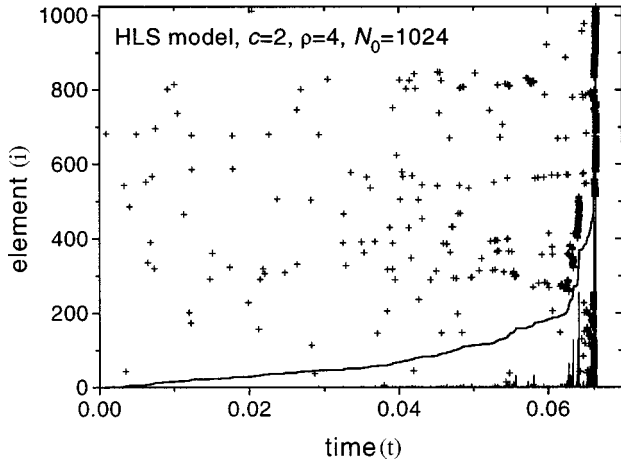


FIG. 1. Accumulated number of failed elements (continuous line) vs time of a hierarchical set. Small crosses represent the position of local fractures. The height of the vertical spikes indicates the load of an element at the time of its failure. Read the text for details.

rate. On the other hand, if one chooses a δ typical of the final stages, it will be so short that although the calculation would be extremely good, at the beginning one would become bored awaiting the outcome of a breaking event. That is to say, these small intervals are not realistic for practical use.

It is for this reason that when one tries to devise an efficient numerical method to accurately describe the time evolution of the system, the choice of the time interval must be adjusted to the characteristic scale at which individual elements break in the process. This characteristic time scale, δ , as mentioned before changes with time, δ_j , and is implemented through the following definition

$$\delta_j = \text{minimum of } \left\{ \frac{1}{N_{l,j} Y_{l,j}^\nu} \right\}, \quad (2.10)$$

where ν is a constant ≥ 1 , independent of l and of j ; we will call it the time resolution parameter. The length of δ_j as defined in Eq. (2.10) points, at each j , to a specific sublist whose l will be denoted as k_j . Now we define a probability of failure for each sublist,

$$p_{l,j} = Y_{l,j}^\nu \delta_j. \quad (2.11)$$

As $Y_{l,j}^\nu$ is the failure rate for elements in sublist l , $Y_{l,j}^\nu \delta_j$ is the expected number of casualties per element in sublist l , i.e., the probability of failure for sublist l . The product $N_{l,j} p_{l,j}$ is maximum for the sublist k_j , in this case $N_{k,j} p_{k,j} = 1/\nu$, which means that when the comparisons below Eq. (2.12) are performed, elements belonging to k_j are the most likely to fail. In particular if $\nu=1$, one element of the k th sublist is likely to fail. For the other sublists, the probability of an individual failure is lower than one. However, any element of any l_j has a nonzero chance of failing in this probabilistic approach. We have called ν the time resolution parameter because if it grows the time intervals δ_j are smaller and therefore it is obvious that the process of failure is more finely resolved. Then the probability $p_{l,j}$ is compared, for each element belonging to the sublist l , with a random number, n , $0 \leq n \leq 1$.

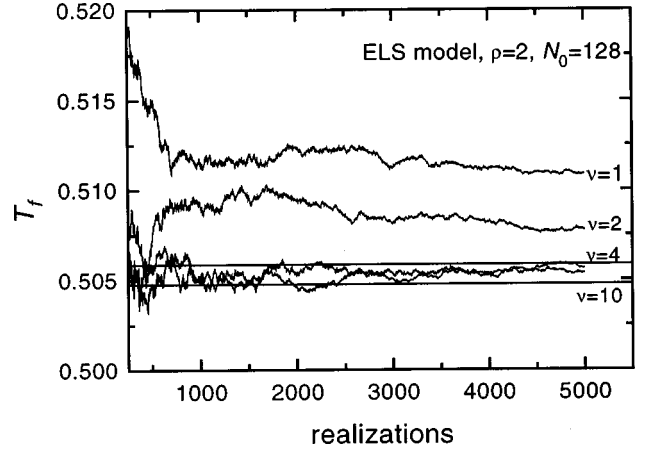


FIG. 2. T_f results using probabilistic method for various ν . The two horizontal lines show the prediction of the usual method.

$$\text{If } p_{l,j} > n \text{ the element fails.} \quad (2.12a)$$

$$\text{If } p_{l,j} < n \text{ the element survives.} \quad (2.12b)$$

The elements that fail in any of the sublists transfer their load according to the rule of transfer and the information contained in the list $\{\bar{x}\}_j$. In the case that no element fails, a new time interval δ_{j+1} equal to δ_j is added and the same probabilities, $p_{l,j+1} = p_{l,j}$, are compared with random numbers. This is repeated until at least one failure occurs that modifies $\{\bar{x}\}_j$, $\{\sigma\}_j$, $\{Y_l, N_l\}_j$, and hence δ_j . The total time to failure, T_f , is the sum of the δ_j up to the disappearance of all the elements.

In the ELS case, δ_j can be explicitly written. After $(j-1)$ steps and assuming one failure per step, the number of surviving elements forming the unique sublist is $N_j = N_0 - (j-1)$, and the individual load is $Y_l = (N_0/N_j)$. Then

$$\delta_j = \frac{1}{N_0 - (j-1)} \left(\frac{N_0 - (j-1)}{N_0} \right)^\rho = \frac{(N_0 + 1 - j)^{\rho-1}}{N_0^\rho}. \quad (2.13)$$

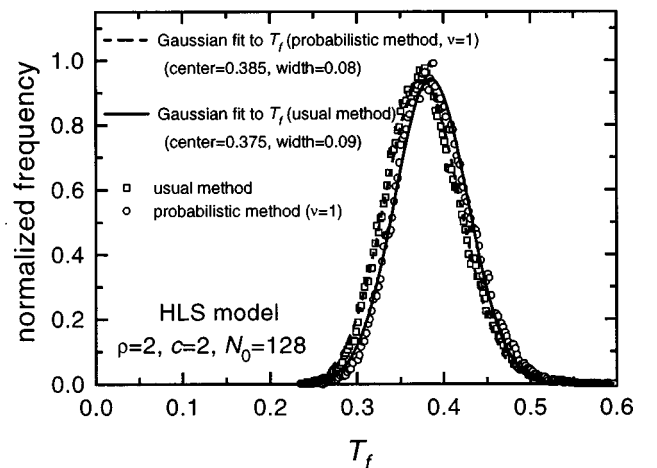
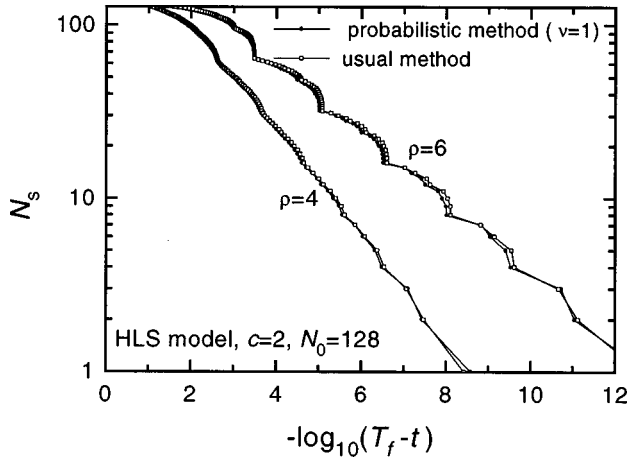


FIG. 3. Comparison of the two methods, for a hierarchical model.

FIG. 4. Evolution of N_s over time for both methods.

Note that we have used $\nu=1$ to be in accordance with the one failure per step assumption. In Eq. (2.13) one observes that, in the first step, $\delta_1=1/N_0$, and in the last step, $\delta_{N_0}=1/N_0^\rho$. Now we proceed to sum up all the time intervals. In the continuous limit, we find

$$T_f = \int_1^{N_0} \frac{(N_0+1-x)^{\rho-1}}{N_0^\rho} dx = \frac{1}{\rho} \left[1 - \frac{1}{N_0^\rho} \right], \quad (2.14)$$

which tends to the correct result $1/\rho$ in the limit of large N_0 .

III. RESULTS AND DISCUSSION

For the ELS case, $\rho=2$, $N_0=100$, we plot in Fig. 2 the average of T_f after the number of simulations expressed in the abscissas, for various ν . The horizontal lines comprise the extremes of the values obtained in 10 averages of 32 000 simulations each by means of the usual method. One can observe, (a) the actual T_f of this set is not $1/\rho$ as predicted by the differential equation, this being a finite-size effect; and (b) $\nu=4$ is already sufficient in this method to reproduce the result of the usual approach. For the HLS case, $\rho=2$ and coordination number of the Cayley tree, $c=2$, we show in Fig. 3 the dispersion of T_f emerging from the usual method (squares) and from the new method (circles) with $\nu=1$. Note the slight shift rightwards of the center of the Gaussian, i.e., the values are longer. This is in agreement with what is seen in Fig. 2. A greater value of ν would move the Gaussian to the left up to the coincidence. In Fig. 4, the averaged rates of breaking of a set of $N_0=128$ are plotted under the HLS rule, $c=2$, for two values of the Weibull exponent $\rho=4$ and $\rho=6$. We compare the habitual method and the new method for $\nu=1$. In Table I, a set of values of T_f and their intrinsic width is shown for the HLS case, $c=2$, by varying N_0 and ν .

TABLE I. Comparison of the two methods (HLS, $c=2$) (see text for details).

Method	$\rho=2$		$\rho=4$	
	$N_0=128^a$	$N_0=512^b$	$N_0=128$	$N_0=512$
Prob($\nu=1$)	0.3890	0.3614	0.0992	0.0750
Prob($\nu=4$)	0.3807	0.3577	0.0907	0.0718
Prob($\nu=10$)	0.3781	0.3573	0.0901	0.0710
Usual	0.3776	0.3565	0.0888	0.0705

^aSimulations with $N_0=128$ elements are averages over 32 000 realizations. Standard deviation of the mean value is ± 2 units in the least significant digit.

^bSimulations with $N_0=512$ elements are averages over 10 000 realizations. Standard deviation of the mean value is ± 1 unit in the least significant digit.

Data for $N_0=128$, and $N_0=512$ are averages over 32 000, and 10 000 realizations, respectively. The errors quoted are one standard deviation of the mean. Perhaps the most abrupt rule of transfer that one can imagine is that of the local one-dimensional unilateral model [14], where the load of failed elements is transferred to the nearest neighbor in the row going in one direction. This implies the almost immediate opening of big cracks and hence a great instability. The probabilistic approach for this model has been tested and again both methods coincide.

Note that in the probabilistic method, there can be δ_j in which no element fails, and others in which several elements do. In contrast to the usual method, here no disorder is fixed at the start: we begin with N_0 elements, all with the same mean life; the random successive failures are responsible for the fluctuations, i.e., this is an example of annealed disorder. For a small value of ν , the results emerging from the probabilistic method are already indistinguishable from those deriving from the usual method. So, we have numerically proved the equivalence of the two approaches. If ν is greater, the method demands more effort but the results reach a saturation point. Comparing the respective disadvantages of computing: in the probabilistic approach it is necessary to deal with larger sets of random numbers while in the usual method the set of stored data is much bigger.

We conclude by quoting Feynman who, in his original paper on path integral formalism [15], writes “although it does not yield new results there is a pleasure in recognizing old things from a new point of view.”

ACKNOWLEDGMENTS

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