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Diluted banded random matrices: scaling behavior of eigenfunction and spectral properties

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Abstract

We demonstrate that the normalized localization length β of the eigenfunctions of diluted (sparse) banded random matrices follows the scaling law $\beta = x^*/(1 + x^*)$. The scaling parameter of the model is defined as $x^* \propto (b_{\text{eff}}^2/N)^{\delta}$, where b_{eff} is the average number of non-zero elements per matrix row, N is the matrix size, and $\delta \sim 1$. Additionally, we show that x^* also scales the spectral properties of the model (up to certain sparsity) characterized by the spacing distribution of eigenvalues.

Keywords: random matrix theory, scaling laws, wigner-banded random matrix model, localization length of eigenfunctions

(Some figures may appear in colour only in the online journal)

1. Introduction and outlook

Random matrix (RM) models serve to describe statistical properties of complex systems and related processes: From the original Gaussian ensembles of Wigner and Dyson [1, 2] (which reproduce the statistics of energy levels of complex nuclei, quantized chaotic systems,

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disordered systems, random networks, etc) to very recent and more elaborated ensembles e.g. relevant to the problem of many-body localization [3].

Even at the early years of RM modeling, Wigner himself realized the need to refine the generic Gaussian ensembles in order to incorporate properties of realistic physical systems. In this respect he introduced the so-called Wigner-banded RM model [4, 5] (see also [2, 6-13]), a model including a bandwidth and an increasing diagonal. In particular the bandwidth, which quantifies the range of interactions, has been the main ingredient of other RM models proposed to deal with explicit applications: As examples we can mention the power-law banded RM model [14, 15] (used to simulate the Anderson metal-insulator transition), the banded random matrix (BRM) model [16–32] (introduced to emulate quasi-one-dimensional disordered wires), the embedded ensembles [33–35] (which take into account the many-body interactions in complex nuclei and many-body systems), system-specific banded Hamiltonian RM models [36, 37] (where the bandwidth of the Hamiltonian matrix can be obtained by means of semiclassical arguments [7, 38]), among many others [2, 39–51].

On the other hand, there exist several works dealing with diluted RM models, see for example [52–65]. However, we know just a few RM models including, in addition to sparsity, an effective bandwidth: i.e. the Wigner-banded RM model with sparsity [47, 48], diluted power-law RM models [49, 50], and a diluted block-banded RM model [51].

Thus, motivated by the ample interest on banded RM models and the recent attention on diluted versions of them [49–51] in this paper we study scaling properties of a diluted version of the BRM model. In particular we demonstrate that both eigenfunction and spectral properties scale with a parameter that relates the model attributes (matrix size, bandwidth, and sparsity) in a highly non-trivial way.

2. Model definition and statement of the problem

The BRM ensemble is defined as the set of $N \times N$ real symmetric matrices whose entries are independent Gaussian random variables with zero mean and variance $1 + \delta_{i,j}$ if |i - j| < b and zero otherwise. Hence, the bandwidth *b* is the number of nonzero elements in the first matrix row which equals 1 for diagonal, 2 for tridiagonal, and *N* for matrices of the GOE. There are several numerical and theoretical studies available on this model, see for example [16–32]. In particular, outstandingly, it has been found [16–18, 21] that the eigenfunction properties of the BRM model, characterized by the *scaled localization length* β (see equation (6) below), are *universal* for the fixed ratio

$$X = \frac{b^2}{N} . \tag{1}$$

More specifically, it was numerically and theoretically shown that the scaling function

$$\beta = \frac{\Gamma X}{1 + \Gamma X},\tag{2}$$

with $\Gamma \sim 1$, holds for the eigenfunctions of the BRM model, see also [22–25]. It is relevant to mention that scaling (2) was also shown to be valid, when the scaling parameter X is properly defined, for the kicked-rotator model [21, 66, 67] (a quantum-chaotic system characterized by a random-like banded Hamiltonian matrix), the one-dimensional Anderson model, and the Lloyd model [68].

We define the *diluted* BRM (dBRM) model by including sparsity, characterized by the parameter α , in the BRM model as follows: Starting with the BRM model we randomly set

off-diagonal matrix elements to zero such that the sparsity is defined as the fraction of the N(b-1)/2 independent non-vanishing off-diagonal matrix elements. According to this definition, a diagonal random matrix is obtained for $\alpha = 0$, whereas the BRM model is recovered when $\alpha = 1$.

Therefore, inspired by scaling studies of the BRM model [16–18, 21–23, 26, 28, 30], here we propose the study of eigenfunction and spectral properties of the dBRM model as a function of the parameter

$$x = \frac{b_{\text{eff}}^2}{N}, \qquad b_{\text{eff}} \equiv \alpha b,$$
 (3)

where the *effective bandwidth* of the dBRM model b_{eff} is, in analogy to the bandwidth b of the BRM model, the *average* number of nonzero elements per matrix row.

3. Eigenfunction properties

A commonly accepted tool to characterize quantitatively the complexity of the eigenfunctions of random matrices (and of Hamiltonians corresponding to disordered and quantized chaotic systems) is the information or Shannon entropy *S*. This measures provides the number of principal components of an eigenfunction in a given basis. The Shannon entropy, which for the eigenfunction Ψ^m is given as

$$S = -\sum_{n=1}^{N} (\Psi_n^m)^2 \ln(\Psi_n^m)^2,$$
(4)

allows to compute the so called entropic eigenfunction localization length, see e.g. [67],

$$\ell_N = N \exp\left[-\left(S_{\text{GOE}} - \langle S \rangle\right)\right],\tag{5}$$

where $S_{\text{GOE}} \approx \ln(N/2.07)$, which is used here as a reference, is the entropy of a random eigenfunction with Gaussian distributed amplitudes (i.e. an eigenfunction of the GOE). With this definition for *S* when $\alpha = 0$ or b = 1, since the eigenfunctions of the dBRM model have only one non-vanishing component with magnitude equal to one, $\langle S \rangle = 0$ and $\ell_N \approx 2.07$. On the other hand, when $\alpha = 1$ and b = N we recover the GOE and $\langle S \rangle = S_{\text{GOE}}$; so, the *fully chaotic* eigenfunctions extend over the *N* available basis states and $\ell_N \approx N$.

Here, as well as in BRM model studies, we look for the scaling properties of the eigenfunctions of the dBRM model through the *scaled localization length*

$$\beta = \frac{\ell_N}{N},\tag{6}$$

which can take values in the range (0, 1].

In the following we use exact numerical diagonalization to obtain the eigenfunctions Ψ^m (m = 1...N) of large ensembles of dBRMs characterized by the parameters *N*, *b*, and α . We perform the average $\langle S \rangle$ taking half of the eigenfunctions, around the band center, of each random matrix. Each average is computed with 5×10^5 data values.

In figure 1(a) we present β as a function of x, see equation (3), for ensembles of matrices characterized by the sparsity α . We observe that the curves of β versus x have a functional form similar to that for the BRM model (corresponding to $\alpha = 1$). In addition, in figure 1(b) the logarithm of $\beta/(1-\beta)$ as a function of $\ln(x)$ is presented. The quantity $\beta/(1-\beta)$ was useful in the study of the scaling properties of the BRM model [16, 22] because $\beta/(1-\beta) = \gamma x$, which is equivalent to scaling (2), implies that a plot of $\ln[\beta/(1-\beta)]$ versus $\ln(x)$ is a straight



Figure 1. (a) Scaled localization length β as a function of $x = b_{\text{eff}}^2/N$ [see equation (3)] for ensembles of diluted banded random matrices characterized by the sparsity α . Horizontal dot-dashed lines at $\beta \approx 0.12$ and 0.88 are shown as a reference, see the text. (b) Logarithm of $\beta/(1-\beta)$ as a function of $\ln(x)$. Upper inset: power δ , from the fittings of the data with equation (7), as a function of α . Lower inset: enlargement in the range $\ln[\beta/(1-\beta)] = [-2, 2]$ including data for $\alpha = 0.6$, 0.8, and 1. Lines are fittings of the data with equation (7).

line with unit slope. Even though, this statement is valid for the BRM model in a wide range of parameters (i.e. for $\ln[\beta/(1-\beta)] < 2$) it does not apply to the dBRM model; see figure 1(b). In fact, from this figure we observe that plots of $\ln[\beta/(1-\beta)]$ versus $\ln(x)$ are straight lines (in a wide range of x) with a slope that depends on the sparsity α . Consequently, we propose the scaling law

$$\frac{\beta}{1-\beta} = \gamma x^{\delta},\tag{7}$$

where both γ and δ depend on α . Indeed, equation (7) describes well our data, mainly in the range $\ln[\beta/(1-\beta)] = [-2, 2]$, as can be seen in the inset of figure 1(b) where we show the numerical data for $\alpha = 0.6$, 0.8 and 1 and include fittings with equation (7). We stress that the range $\ln[\beta/(1-\beta)] = [-2, 2]$ corresponds to a reasonable large range of β values, $\beta \approx [0.12, 0.88]$, whose bounds are indicated with horizontal dot-dashed lines in figure 1(a). Also, we notice that the power δ , obtained from the fittings of the data using equation (7), is very close to unity for all the sparsity values we consider here (see the upper inset of figure 1(b)).

Therefore, from the analysis of the data in figure 1, we are able to write down a *universal* scaling function for the scaled localization length β of the dBRM model as

$$\frac{\beta}{1-\beta} = x^*, \qquad x^* \equiv \gamma x^\delta . \tag{8}$$

To validate equation (8) in figure 2(a) we present again the data for $\ln[\beta/(1-\beta)]$ shown in figure 1(b) but now as a function of $\ln(x^*)$. We do observe that curves for different values of α fall on top of equation (8) for a wide range of the variable x^* . Moreover, the collapse of the numerical data on top of equation (8) is excellent in the range $\ln[\beta/(1-\beta)] = [-2, 2]$ for $\alpha \ge 0.5$, as shown in the inset of figure 2(a).

Finally, we rewrite equation (8) into the equivalent, but explicit, scaling function for β :



Figure 2. (a) Logarithm of $\beta/(1-\beta)$ as a function of $\ln(x^*)$ [see equation (8)]. Inset: enlargement in the range $\ln[\beta/(1-\beta)] = [-2, 2]$ including curves for $\alpha \in [0.5, 1]$ in steps of 0.05. Dashed lines in main panel and inset are equation (8). (b) β as a function of x^* . Inset: data for $\alpha \in [0.5, 1]$ in steps of 0.05. Dashed lines in main panel and inset are equation (9).

$$\beta = \frac{x^*}{1+x^*} \,. \tag{9}$$

In figure 2(b) we confirm the validity of equation (9). We would like to emphasize that the universal scaling given in equation (9) extends outsize the range $\beta \approx [0.12, 0.88]$, for which equation (7) was shown to be valid, see the main panel of figure 2(b). Furthermore, the collapse of the numerical data on top of equation (9) is remarkably good for $\alpha \ge 0.5$, as shown in the inset of figure 2(b).

4. Spectral properties

Now we analyze the spectral properties of the dBRM model. To this end we choose P(s), the *nearest neighbor energy-level spacing distribution*. For $\alpha = 0$ or b = 1, i.e. when the dBRM model produces diagonal matrices, P(s) follows the exponential distribution $P(s) = \exp(-s)$; better known in RM theory as Poisson distribution or the spacing rule for random levels [1]. In the opposite limit, for $\alpha = 1$ and b = N, i.e. when the dBRM reproduces the GOE, P(s) closely follows the Wigner–Dyson distribution [1]: $P(s) = (\pi/2)s \exp(-\pi s^2/4)$. Then, for values of α and b in the intervals (0, 1) and (1, N), respectively, the shape of the P(s) of the dBRM model is expected to be in-between Poisson and Wigner–Dyson distributions. Moreover, a transition from Poisson to Wigner–Dyson distributions in the shape of P(s) is expected in the following two cases: (i) By increasing α from zero to one for fixed b, with $1 \ll b \le N$; and (ii) by increasing b from one to N for fixed α , with $0 < \alpha \le 1$. These two scenarios are reported in figures 3(a) and (b), respectively.

Here, in order to characterize the P(s) for our RM model we use the phenomenological expression known as Izrailev's distribution [26, 69]:

$$P(s) = B_1 z^{\widetilde{\beta}} (1 + B_2 \widetilde{\beta} z)^{f(\widetilde{\beta})} \exp\left[-\frac{1}{4} \widetilde{\beta} z^2 - \left(1 - \frac{\widetilde{\beta}}{2}\right) z\right],$$
(10)



Figure 3. Nearest-neighbor energy level spacing distribution P(s) for ensembles of diluted banded random matrices. N = 1000 in all cases. In (a) b = 212 while α takes values from 0.001–0.1. In (b) $\alpha = 0.7$ while *b* takes values from 3–52. Blue and red full lines correspond to Poisson and Wigner–Dyson distribution functions, respectively. Dashed lines are fittings of the histograms with Izrailev's distribution of equation (10), where the fitted values of $\tilde{\beta}$ are (a) $\tilde{\beta} = 0.0795$ (0.077, 0.081) (for $\alpha = 0.005$), $\tilde{\beta} = 0.407$ (0.401, 0.413) (for $\alpha = 0.0075$), $\tilde{\beta} = 0.689$ (0.683, 0.696) (for $\alpha = 0.01$); and (b) $\tilde{\beta} = 0.032$ (0.031, 0.033) (for b = 5), $\tilde{\beta} = 0.221$ (0.218, 0.224) (for b = 13), $\tilde{\beta} = 0.655$ (0.643, 0.662) (for b = 26), where the 95% confidence bounds are given in parenthesis.

where $z = \pi s/2$, $f(\tilde{\beta}) = \tilde{\beta}^{-1} 2^{\tilde{\beta}} (1 - \tilde{\beta}/2) - 0.16874$, and the parameters $B_{1,2}$ are determined by the normalization conditions $\int_0^\infty P(s) ds = \int_0^\infty sP(s) ds = 1$. We call $\tilde{\beta}$ the spectral parameter. Since Izrailev's distribution has been shown to be useful to characterize the P(s) of the BRM model [26], we expect it can properly describe the P(s) of the dBRM model. Indeed, in figure 3 we already show that histograms of P(s) in the transition from Poisson to Wigner–Dyson are well fitted by equation (10), making $\tilde{\beta}$ a good quantity to characterize such transition.

Thus, we construct histograms of P(s) for a large number of combinations of the parameters of the dBRM model (α, b, N) and by fitting them with equation (10) we extract systematically the corresponding values of $\tilde{\beta}$. We always construct P(s) from half of the total unfolded [1] spacings $s_m = (E^{m+1} - E^m)/\Delta$ around the band center, where the density of states is approximately constant. Here, E^m is the *m*th eigenvalue and Δ the mean level spacing. Each histogram is constructed with 5×10^5 spacings.

In figure 4(a) we present the spectral parameter β as a function of the scaled localization length β for the dBRM model. As in figures 1 and 2, here we label different sparsities α with different colors (symbols). It is interesting to note that even though the relation between $\tilde{\beta}$ and β is not simple, e.g. linear as reported for other disordered systems [9, 67, 70–72], the curves $\tilde{\beta}$ versus β are independent of α once $\alpha > 0.4$, see inset of figure 4(a). This allows us to guess that x^* can also serve to scale the spectral parameter $\tilde{\beta}$, at least for $\alpha > 0.4$. Accordingly, in figure 4(b) we show that the curves of $\tilde{\beta}$ versus x^* fall one on top of the other mainly for $\alpha > 0.4$, see the inset of the figure.

From figure 4(b) we also observe that the curves $\tilde{\beta}$ versus x^* are above equation (9), that we include as dashed lines, except for very small values of x^* where they coincide. This fact has



Figure 4. (a) Spectral parameter $\tilde{\beta}$ (see equation (10)) as a function of the scaled localization length β for ensembles of diluted banded random matrices characterized by the sparsity α . Inset: $\ln \tilde{\beta}$ versus $\ln \beta$ for $\alpha \in [0.5, 1]$ in steps of 0.05. Dashed lines in main panel and inset are the identity. (b) Repulsion parameter $\tilde{\beta}$ as a function of x^* . Inset: data for $\alpha \in [0.5, 1]$ in steps of 0.05. Dashed lines in main panel and inset are equation (9).

already been reported for the BRM model in [21]. This also means that the spectral properties of the dBRM model approach the GOE limit faster than the eigenfunction properties.

5. Conclusions

In this paper, by using extensive numerical simulations, we demonstrate that the normalized localization length β of the eigenfunctions of a diluted banded random matrix (dBRM) model scales with the parameter $x^*(N, b, \alpha) = \gamma(\alpha)[(b\alpha)^2/N]^{\delta(\alpha)}$ as $x^*/(1 + x^*)$, where (N, b, α) are the model parameters (matrix size, bandwidth, and sparsity, respectively) and γ and δ are scaling parameters. In addition, by plotting the spectral parameter $\tilde{\beta}$ (the repulsion parameter of Izrailev's distribution) as a function of β we realized that, for moderate sparsity (i.e. $\alpha > 0.4$), $x^*(N, b, \alpha)$ also scales the spectral properties of the dBRM model.

While general diluted RM models have direct applications to random networks (i.e. the adjacency matrices of complex networks are, in general, diluted random matrices), the dBRM model may be used to model multilayer random networks since the bandwidth b and the sparsity α can be associated, respectively, to the size and connectivity of the subnetworks composing a multilayer, see e.g. [51].

Finally, we want to recall that the scaling (2), valid for the BRM model (and other disordered systems [66, 68]), was rewritten in a more elegant way as a relation between properlydefined inverse lengths [22, 22]

$$\frac{1}{d(N,W)} = \frac{1}{d(\infty,W)} + \frac{1}{d(N,0)},\tag{11}$$

were $d(N, W) \equiv \exp[\langle S(N, W) \rangle]$ and *W* represents *b* for the BRM model or the localization length for the one-dimensional Anderson model and Lloyd's model. Here, in the case of the dBRM model, scaling (9) can also be written in the 'model independent' form (11) as

$$\frac{1}{d(N,b,\alpha)} = \frac{1}{d(\infty,b,\alpha)} + \frac{1}{d(N,N,1)},$$
(12)

with $d(N, b, \alpha) \equiv \exp[\langle S(N, b, \alpha) \rangle]$ and $d(N, N, 1) = \exp[S_{\text{GOE}}(N)]$ (the reference entropy). We hope our results may motivate a theoretical approach to the dBRM model.

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