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Multilayer Networks in a Nutshell

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Abstract

Complex systems are characterized by many interacting units that give rise to emergent behavior. A particularly advantageous way to study these systems is through the analysis of the networks that encode the interactions among the system constituents. During the past two decades, network science has provided many insights in natural, social, biological, and technological systems. However, real systems are often interconnected, with many interdependencies that are not properly captured by single-layer networks. To account for this source of complexity, a more general framework, in which different networks evolve or interact with each other, is needed. These are known as multilayer networks. Here, we provide an overview of the basic methodology used to describe multilayer systems as well as of some representative dynamical processes that take place on top of them. We round off the review with a summary of several applications in diverse fields of science.

1. INTRODUCTION

In 1972, Philip W. Anderson published his famous article *Why More Is Different*, where he criticized the reductionist hypothesis according to which everything, from matter to life, could be completely described from the same set of fundamental laws (1). The fallacy, he said, was that a reductionist approach did not imply a constructionist one. That is, being able to reduce something to its fundamental laws does not imply the ability to start from them and reconstruct it. This simple, yet powerful, idea is the key concept behind what we today call complex systems.

Complex systems are defined as those systems exhibiting emergent phenomena. The definition of emergence is, however, still a subject of intense debate. Given the scope of this review, we settle for one of the most common definitions: Emergence is related to phenomena that cannot be explained or predicted just from the constituent parts of the system (nevertheless, we refer the interested reader to Falkenburg & Morrison (2) for more information). This definition of emergence implies a hierarchical view of the world, which calls again into question the role of reduction.

A simple example of the limitations of a reductionist approach is water, as its properties cannot be predicted from those of hydrogen and oxygen on their own (3). If we do not consider them isolated but allow them to interact, the atoms will establish chemical bonds to form molecules from whose interactions we can extract most properties of water. Thus, to correctly address some systems a more holistic view is needed, in which we account for both its constituents and their interactions.

The main framework used to encode these characteristics of systems in a mathematically tractable object is graph theory. The application of statistical physics to graphs was initially aimed at studying problems like percolation processes. However, as interest in complex systems increased, it was promptly discovered that, with the addition of techniques borrowed from other scientific disciplines such as computer science or statistical inference, it was possible to go beyond classical physics systems and study the behavior of a much broader array of problems in fields as diverse as economic, social, and biological sciences. The combination of these techniques provided us with a very powerful tool to study complex systems that is usually known as network science.

In its most basic form, a network reduces a system to an abstract structure consisting of some simplified entities, called nodes, which retain some of the properties of the original components of the system, and the connection patterns between them, their links. This representation not only allows for a more complete view of the system but also uncovers some novel characteristics. In particular, the structure of the networks, that is, the particular pattern of interactions of its components, can have a big effect on the behavior of the system (4).

Nonetheless, this mathematical representation might still be a reduction of the original system. Thus, one may wonder whether we could end up in the same reductionism that we wanted to avoid, albeit at a different level of description. Network-based approaches have produced significant insights into the structure and dynamics of complex systems during the past two decades or so. However, it is possible to add more information to these models while keeping them tractable and making them more powerful at the same time. This is done by classifying the interactions found in a system into groups according to their characteristics. This classification yields a set of networks, one for each type of interaction, connected to each other. The way in which these networks are connected to each other, the entities that nodes represent, and the way their interactions are represented produces a new set of networks that goes beyond the concept of simple graphs; these are multilayer networks (5).

2. FROM SIMPLE GRAPHS TO MULTILAYER NETWORKS

The formalism of multilayer networks is an extension of the one for single networks (henceforth also called monoplex), i.e., networks that are completely described by just a set of entities (nodes)

and their interactions (links). For this reason, we begin this discussion by providing a brief introduction to the notion of single graphs, and then we build the multilayer network framework from there. Nevertheless, we refer the interested reader to books by Newman (4) and Barabsi (6) for further details.

2.1. Networks and Graphs

In their most general form, networks are mathematically represented by graphs. A graph is a tuple $G = (V, E)$, where V is the set of nodes (entities of the system) and $E \in V \times V$ is the set of links that connect pairs of nodes (two nodes are connected if they interact in some way). If E is an unordered set, the graph is said to be undirected, whereas if it is an ordered set, the graph is directed. In this last case, changes in the node where the link starts (source) will affect the node where it ends (target), but changes in the target node will not directly affect the source node. Note, however, that as we are treating the system as a whole, it may produce effects on the source node via other nodes. Additionally, in some cases, links have an associated number called weight, which represents the intensity of the interaction.

Nonetheless, the usual representation of networks is the adjacency matrix. The adjacency matrix $\mathcal{A}$ of a monoplex undirected graph is an $N \times N$ symmetric matrix (where N is the number of nodes in the network) with elements a_{ij} such that $a_{ij} = 1$ if there is an edge between nodes i and j (or w if the link has an associated weight) or $a_{ij} = 0$ otherwise. Conversely, if the monoplex is directed the matrix would not be symmetric anymore, and a_{ij} would be 1 only when the edge goes from i to j.

The most basic measure in networks is the degree of a node, which is defined as the number of links it possesses,

$$k_i = \sum_{j=1}^{N} a_{ij}.$$

We define the degree distribution of a network as the set of quantities p_k, which represent the fraction of nodes with degree k. It is possible to characterize networks by their degree distribution, and almost all real-world networks have degree distributions with a tail of high-degree nodes.

The structural properties of the networks provide us a lot of information about their dynamics. For example, suppose that in the node i there is an amount ψ_i of some substance that is able to move along the links. The rate at which this substance flows from node j to i can be written as $C(\psi_j - \psi_i)$, where C is a diffusion constant. Then, the rate at which ψ_i is changing is given by

$$\frac{\mathrm{d}\psi_i}{\mathrm{d}t} = C \sum_j a_{ij}(\psi_j - \psi_i).$$

This equation can be rewritten as $\frac{\mathrm{d}\psi}{\mathrm{d}t} + C\mathcal{L}\psi = 0$, where $\psi = \{\psi_i\}$. $\mathcal{L}$ is called the graph Laplacian and is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D}$ is a diagonal matrix whose entry i is k_i. The interesting aspect about this matrix is that its eigenvalues completely describe the diffusion process.

Another important concept in network science is the notion of path. A path is a sequence of nodes such that every consecutive pair of them is connected by a link in the network. A particularly interesting example of path is the random walk. A random walk is a path in which each step is chosen uniformly at random from the set of links attached to the current node. It can be shown that in the long time limit the probability of finding the random walk at node i is proportional to its degree, $p_i = \frac{k_i}{2m}$, where m is the number of links in the network.

In the same way that the eigenvalues of the adjacency and Laplacian matrices that describe the network can explain some of its dynamics, there are other measures of the structure of networks that

are very useful for understanding their behavior. One of the most interesting structural measures are those related to the concept of centrality. Centrality, in this context, refers to the importance of a node in the network, although the definition of importance can vary from one system to another. The degree itself can be regarded as a measure of centrality, known as degree centrality, as nodes with several links can produce effects in large groups of nodes. However, sometimes a node can be considered important not because of its intrinsic importance but because it is linked to other nodes that are themselves important. This is the concept behind eigenvector centrality, which can be obtained from the eigenvector corresponding to the largest eigenvalue of the adjacency matrix. Another interesting centrality measure is the betweenness centrality, which is defined as the fraction of shortest paths between nodes s and t that pass through node i out of the total number of shortest paths between those nodes. Thus, this quantity measures the extent to which a vertex lies on paths between other nodes.

Another interesting property of networks is that there are also intermediate scales that might play a key role for the functioning of the system at the global level. Not only are individual nodes important but it is also possible to find groups of nodes that share common properties. The simplest one is the group formed by three nodes. If all of them are connected to all others, the relationship is said to be transitive. To quantify the level of transitivity in a network, we define the clustering coefficient as the fraction of paths of length three in the network that are closed. Similarly, we might be able to classify nodes into groups or classes. If nodes of a given class tend to connect to nodes belonging to the same class, we say that the network is assortative. It is usually useful to measure whether the assortativity is larger than expected if nodes were randomly connected. Finally, another important structural property is related to whether or not so-called communities or modules can be identified in a network. A community or module is a set of nodes that are more densely connected among themselves than they are with nodes that are not in the same set (or module). There are several methods and heuristics for community detection, but the most popular ones are those based on maximizing the modularity of the network, which is defined as

$$Q = \frac{1}{2m} \sum_{ij} \left(a_{ij} - \frac{k_i k_j}{2m} \right) \delta(c_i, c_j),$$

where c_i is the class i belongs to.

Up to now, we have considered that the networks under study are isolated, which is equivalent to assuming that nodes are connected to each other by a single type of link and that other possible interactions not considered in the set E are neglected. However, in many cases, this assumption can be an oversimplification. For example, neglecting time dependence destroys the ordering of transmission processes. Similarly, ignoring the presence of multiple types of links might introduce errors when dynamical processes depend on the type of interaction. Thus, if we want to progress deeper in the study of complex systems, we need to move to more advanced network structures, i.e., multilayer networks.

2.2. Multilayer Networks

To represent systems consisting of networks with multiple types of links, or other similar features, we consider structures that have layers in addition to nodes and links. In its more general form, a multilayer network has a node u in layer α that can be connected to any node v in any layer β. Layers represent aspects or features that characterize the nodes or the links that belong to that layer. Consequently, we can partition the set of links into intralayer links, that is, links that connect nodes set in the same layer, and interlayer or coupling links, which are those links that connect nodes set in different layers.

Following De Domenico et al. (7), we use tensors, as a generalization of matrices, to represent these multilayer objects. We can define the intralayer adjacency tensor $W^{\alpha}_{\beta}(\tilde{k}) = \sum_{i,j=1}^{N} w_{ij}(\tilde{k}) E^{\alpha}_{\beta}(ij)$ as the one that indicates the relationships between nodes within the same layer $\tilde{k}$. Note that in unweighted monoplex networks $w_{ij}(\tilde{k}) = a_{ij}$, which are the entries of the adjacency matrix. From now on, we differentiate indices that correspond to nodes from those that correspond to layers by using a tilde symbol.

To incorporate the possibility of a node in one layer $\tilde{h}$ being connected to another one in layer $\tilde{k}$, we introduce the interlayer adjacency tensor $C^{\alpha}_{\beta}(\tilde{h}\tilde{k})$, which corresponds to $W^{\alpha}_{\beta}(\tilde{k})$ if $\tilde{h} = \tilde{k}$. Lastly, by denoting the canonical basis as $E^{\tilde{\gamma}}_{\tilde{\delta}}(\tilde{h}\tilde{k}) = e^{\tilde{\gamma}}(\tilde{h}) e_{\tilde{\delta}}(\tilde{k})$, we can write the multilayer adjacency tensor as

$$M^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}} = \sum_{\tilde{h},\tilde{k}=1}^{L} C^{\alpha}_{\beta}(\tilde{h}\tilde{k}) E^{\tilde{\gamma}}_{\tilde{\delta}}(\tilde{h}\tilde{k}),$$

where L is the number of layers. Alternatively, it is possible to extend the adjacency-matrix representation to encode multilayer networks, making it easier to use tools that have already been developed to study monoplex networks in this new framework. We can do so by building a block matrix in which each diagonal block corresponds to the adjacency matrix of each layer, A^{α}, and the off-diagonal matrices encoding the coupling, $C_{\alpha\beta}$ (see **Figure 1**).

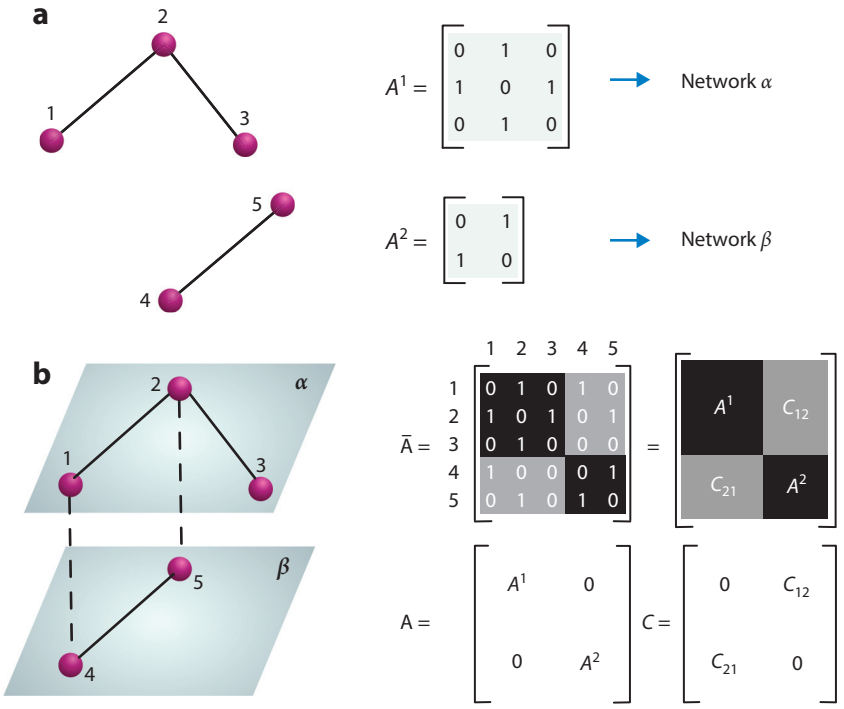

Figure 1

The figure shows (*a*) monoplex networks and (*b*) a multilayer system made up of the two monoplex networks in panel *a*. (*a*) The interactions between the first set of nodes (1,2,3) are not affected and do not affect the second set of nodes (4,5). Therefore, each system corresponds to a single-layer network, whose adjacency matrices are shown. (*b*) When the sets of nodes are not independent and they affect each other, a multilayer representation is more accurate. In the example shown, the structure of each layer is represented by an adjacency matrix $\mathcal{A}^{(i)}$, where $i = \{\alpha = 1, \beta = 2\}$. $\mathcal{C}_{(\alpha\beta)}$ stores the connections between layers α and β. Note that the number of nodes in each layer is not the same.

Multilayer networks encode two major classes of systems (8): multiplex networks and a network of networks. Multiplex networks are networks in which the same set of nodes is represented in every layer, although the interaction between nodes might be different in each one. As an example, two nodes might be connected in one layer and might not in other. This is the case of online social systems, in which a given user might have a Twitter account (layer 1) and a Facebook profile (layer 2). The set of followers/friends does not in general coincide for both layers, thus leading to two different intralayer adjacency matrices. On the contrary, a network of networks is instead formed by networks that are interlaced to each other but formed by different types of nodes. Although these two are the most common kinds of multilayer networks, Kivelä et al. (5) present several other types and denominations of multilayer networks.

When building multilayer networks it is often not completely clear how to define each layer, the interactions among them, or even how many of them are necessary. De Domenico et al. (9) tackle the problem of reducibility, i.e., defining the number of layers a multilayer network needs to have to accurately represent the structure of the system. Using the Von Neumann entropy on a multilayer network, it is possible to determine whether the multilayer representation is distinguishable from the aggregated network. This way, De Domenico et al. propose that if the aggregation of two layers does not result in a decrease of the relative entropy with respect to the multiplex where they are separated, they should be kept aggregated. Likewise, Menichetti et al. (10) propose a measure of the amount of additional information that can be extracted from multiplex networks over the one contained in the individual layers separately based on the entropy of multiplex ensembles. Furthermore, Kleineberg et al. (11) show that multiplex networks are not just random combinations of single network layers but rather they possess significant hidden geometric correlations. Finally, Cozzo et al. (12) show that as a function of the coupling between layers, different multiplexity regimes can be identified from the spectral properties of the graph.

The extension of the tools already developed for single-layer networks to the multilayer framework is sometimes straightforward (13). For example, we can define the degree of node i in layer α as $k_i^\alpha = \sum_j a_{ij}^\alpha$. Consequently, its degree in the multilayer network is no longer a scalar but the vector $k_i = \{k_i^1, \ldots, k_i^L\}$, which results in a total degree or degree overlap of $o_i = \sum_\alpha k_i^\alpha$. It is also possible to create new measures like the edge overlap, $o_{ij} = \sum_\alpha a_{ij}^\alpha$, which accounts for the number of layers in which the same link exists. Besides, as nodes are now characterized by vectors instead of scalars, we need to develop new tools to simplify their description. For instance, we can quantify the distribution of the degree of a node among the various layers using the entropy of the multiplex degree,

$$H_i = -\sum_{\alpha=1}^{L} \frac{k_i^\alpha}{o_i} \ln\left(\frac{k_i^\alpha}{o_i}\right),$$

or the participation coefficient (14).

There are other measures similar to the edge overlap that only work under the multilayer framework; for instance, there is the interdependence (14). The interdependence of node i is defined as $\lambda_i = \sum_{i \neq j} \frac{\psi_{ij}}{\sigma_{ij}}$, where σ_{ij} is the total number of shortest paths between nodes i and j, and ψ_{ij} is the number of shortest paths between node i and node j that makes use of links in two or more layers. Thus, the interdependence measures how dependent a node is on the multiplex structure in terms of reachability. Equivalently, it is possible to extend this definition from nodes to layers to account for the importance of a given layer in the whole system (15).

Cozzo et al. (16) extended the notion of triadic relations, the transitive relationship mentioned previously, to multiplex networks. They distinguish five different triadic relations: one where the three links are in the same layer, three where two links are in one layer and the third is in a different layer, and finally one where the three links are in three different layers.

Similarly, to study assortativity in multilayer networks (the tendency of nodes to be connected to others that are alike), there are several possibilities: We can analyze the assortativity between nodes in different layers, between layers themselves, or inside each layer separately, etc. De Arruda et al. (17) proposed that if the multilayer adjacency tensor is expressed as $M^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}} = \sum_{\tilde{h},\tilde{r}}^{L} C^{\alpha}_{\beta}(\tilde{h}\tilde{r}) E^{\tilde{\delta}}_{\tilde{\gamma}}(\tilde{h}\tilde{r})$, where $C^{\alpha}_{\beta}(\tilde{h}\tilde{r})$ is the adjacency matrix for layer $\tilde{r}$ if $\tilde{h} = \tilde{r}$ or a coupling matrix otherwise, the assortativity coefficient can be written as

$$\rho(\mathcal{W}^{\alpha}_{\beta}) = \frac{\mathcal{M}^{-1}\mathcal{W}^{\alpha}_{\beta} Q^{\beta} Q_{\alpha} - [0.5\mathcal{M}^{-1}(\mathcal{W}^{\alpha}_{\beta} Q_{\alpha} u^{\beta} + \mathcal{W}^{\alpha}_{\beta} Q^{\beta} u_{\alpha})]^2}{\mathcal{M}^{-1}(\mathcal{W}^{\alpha}_{\beta}(Q_{\alpha})^2 u^{\beta} + \mathcal{W}^{\alpha}_{\beta}(Q^{\beta})^2 u_{\alpha}) - [0.5\mathcal{M}^{-1}(\mathcal{W}^{\alpha}_{\beta} Q_{\alpha} u^{\beta} + \mathcal{W}^{\alpha}_{\beta} Q^{\beta} u_{\alpha})]^2}.$$

This equation is quite versatile as it allows us to study various types of assortativity by just adjusting the value of $\mathcal{W}^{\alpha}_{\beta}$. Indeed, it is possible to study the assortativity in each layer, with $\mathcal{W}^{\alpha}_{\beta} = C^{\alpha}_{\beta}(\tilde{r}\tilde{r})$, the assortativity of the projected network; with $\mathcal{W}^{\alpha}_{\beta} = M^{\alpha\tilde{\delta}}_{\beta\tilde{\gamma}} U^{\tilde{\gamma}}_{\tilde{\delta}}$, the assortativity of selected sets of layers; with $\mathcal{W}^{\alpha}_{\beta}(\mathcal{L}) = M^{\alpha\tilde{\delta}}_{\beta\tilde{\gamma}} \Omega^{\tilde{\gamma}}_{\tilde{\delta}}(\mathcal{L})$, where $\Omega^{\tilde{\gamma}}_{\tilde{\delta}}(\mathcal{L})$ is the unity tensor when $\tilde{\gamma}$ and $\tilde{\delta}$ are selected and zero otherwise; or the correlation between different layers with $\mathcal{W}^{\tilde{\gamma}}_{\tilde{\delta}} = M^{\alpha\tilde{\gamma}}_{\beta\tilde{\delta}} U^{\beta}_{\alpha}$. For the proper definition of $\mathcal{M}^{-1}$ and Q^{α}, we refer the reader to Reference 17. We also note that there are several other types of correlations that can be analyzed in complex networks, see, for example, Nicosia et al. (18).

Following our previous discussion, random walks are often used to explore a network using only local information due to their simplicity. Particularly interesting are biased random walkers whose transition probabilities depend on the topological properties of the destination node, as they can be used to define centrality measures, identify communities, or provide optimal exploration of a network. However, it is not straightforward to extend the notion of random walks from simple networks to multilayer networks. Battiston et al. (19) explore the consequences of redefining random walks in multiplex networks. Note that the richness of multilayer networks in this context comes from the fact that node properties are no longer scalar but vectorial, like their degree, which allows defining more complex biasing functions.

Likewise, random walks can be used to define centrality measures in multilayer networks (20). Other centrality measures used in single-layer networks can also be extended to the framework of multilayer networks like PageRank (21), betweenness (22), or eigenvector centrality (23, 24). As De Domenico et al. (25) showed, when one computes centrality measures in the context of multilayer networks, it is possible to find versatile nodes that cannot be extracted directly from the aggregated network. Furthermore, it is possible to use dynamical processes to characterize the relative position of nodes in multiplex networks (26).

As briefly discussed previously, the spectral properties of networks play a very important role in the dynamics they represent. Cozzo et al. (27), using perturbation theory, showed that the largest eigenvalue of the multiplex network is equal to the one of the adjacency matrix of the dominant layer of the system at a first-order approximation. Similarly, Sánchez-García et al. (28) studied the eigenvalues of the Laplacian in the context of multilayer networks. They showed that the eigenvalues of the quotient (a coarsening of the original network) are interlaced with the eigenvalues of its parent network. This fact has deep consequences as, for example, the relaxation time on a multiplex network is smaller or equal to that of the aggregated network, which in turn can result in faster diffusion processes on multiplex networks than in their aggregated counterparts.

Finally, we examine the state of community detection in multilayer networks. Recall that in some networks nodes tend to cluster into groups giving rise to nontrivial structures. Mucha et al. (29) generalized the concept of modularity considered previously in relation to multilayer networks, which can be used to detect communities not only in static networks but also in time-dependent networks by setting the state of the network at each time window in one layer. There

are other extensions of modularity (30) as well as other methods of community detection like flow-based processes (31) or random walks (32, 33). Methods have also been developed to detect overlapping communities (34) and to study the detectability (35) of community structures in stochastic block models (36–38). Next, we summarize the main results obtained so far for several paradigmatic dynamical processes when they take place on top of multilayer networks.

3. DYNAMICAL PROCESSES ON MULTILAYER NETWORKS

This section is divided in two subsections. First, we explore the problem of percolation, which is one of the first phenomena that was tackled by condensed matter physicists. Even though its study started in the 1940s and there is a large theoretical framework already built, it is still very relevant today owing to its importance in the description of networks. Indeed, not only can it be used to analyze the resilience of networks, which would be the most straightforward application of the theory, but it can also help study problems found in very different fields, for instance, vaccination and herd immunity, which are two relevant applications in health sciences (4). Then, we round off this section with the description of diffusion processes in multilayer networks. We also provide some information about problems in which diffusion processes on networks are quite relevant, like the cases of epidemic and opinion spreading.

3.1. Percolation and Multilayer Networks

In undirected single-layer networks, a connected component is the maximal set of nodes that are all connected to one another via some path. In multilayer networks, this definition can be trivially extended by including links that connect layers, so that the set of nodes is connected to each other within each layer and also connected to the rest of the layers (39). Even though the extension is simple, the incorporation of more layers to the system can make the number of nodes grow quite fast. Thus, it is important to implement new algorithms for the detection of connected components in multilayer networks (40, 41).

One of the main applications of percolation theory in networks is the study of robustness. Robustness in this context refers to the ability of preserving the structure of the network when it is subject to failures or attacks, either in the nodes or in the links. Although the robustness of single-layer networks has been deeply analyzed and the main questions are already answered, in the case of multilayer networks there are new and exciting challenges.

For example, in contrast to single-layer networks in which under random removal of nodes the size of the largest connected component has a continuous transition, Baxter et al. (42) showed that the transition in multiplex networks is hybrid. Below the critical point there is a discontinuity like a first-order transition, but above it, the systems exhibit critical behavior typical of a second-order transition. Similarly, another property that behaves differently when comparing robustness in single-layer networks and multilayer networks is clustering (43). In single-layer networks the effect of clustering in the robustness of the network is very low. However, Shao and colleagues showed that clustering substantially reduces the robustness of some multilayer networks (44, 45).

Buldyrev et al. (46) studied an interdependent network of power grids and computers. In this context, we can regard interlayer links as dependency links. That is, for a node to function in one layer, it requires support from another node that is in a different layer so that it depends on the interlayer link. In this case, the shutdown of power stations led to the failure of nodes in the communication network, which in turn caused further breakdown of power stations. They modeled this process with an interdependent network with two layers and showed that these networks can be less robust than single networks. Indeed, a very important node in one layer

might be connected to a smaller, more fragile node in the other. If this last node disappears, it will radically affect the first layer as it will shut down the node that is important.

There are multiple extensions for these models of failures in interdependent networks. For instance, Min et al. (47) addressed the problem of when the failure of a node in one layer does not directly cause a failure in a node in the other layer but instead depletes the resources that the node gets. If that value gets below a certain threshold, then the node fails. With this model, they found that the system exhibits hysteresis, which would produce an increase in the cost of the recovery process. In a similar way, Shao et al. (48) built networks in which nodes are connected by more than one link to other layers so that the failure of just one node does not always produce breakdowns in the other layer. Conversely, Son et al. (49) used an epidemic spreading approach to address the problem of percolation in interdependent uncorrelated networks.

These processes in interdependent networks are also known as cascading processes. The failure of a node in layer A will shut down some nodes in layer B. Consequently, this will produce even more failures in layer A. This process will continue iteratively until all affected nodes are removed from the network. We also note that, from a numerical point of view, the algorithms used for these models can sometimes be too slow, but there are new algorithms proposed to handle these kinds of problems specifically in multilayer networks (50).

Interdependent networks are not the only networks to have been addressed. Other authors like Zhao et al. (51) have explored the robustness of multiplex networks when their nodes are removed, either by random failures or by targeted attacks. Besides, it is also very important to determine the effects that each network property has on the robustness of networks. For instance, Cellai et al. (52) proposed a percolation model to handle multiplex networks with link overlap based on message passing. Min et al. (53) analyzed the problem of network robustness when there are interlayer degree correlations, and Lee et al. (54) addressed the problem in the context of layers in the international trade network.

Finally, we conclude this section by giving some examples of applications of these percolation processes to networks in very different fields. Reis et al. (55) developed a theory to study the stability of interconnected networks and demonstrated it in the context of brain networks. Brummitt et al. (56) studied bankruptcy spreading using cascades in multiplex financial networks. Finally, Baggio et al. (57) quantified the robustness of socioecological systems under plausible scenarios of change and found that changes in social relations precipitated steeper declines in network interconnectedness than did changes in ecological resources.

3.2. Diffusion Processes in Multilayer Networks

Diffusion processes have long been studied in the context of networks. It has been shown that stochastic processes such as evolutionary games or epidemic spreading present very different behaviors when applied to populations structured in networks versus populations that have no underlying structure. For instance, the epidemic threshold, which is a key quantity in mathematical epidemiology, vanishes under some network configurations. Thus, it is a must to know the underlying structure of the population in which the process is taking place. Besides, as the structure of multilayer networks encodes new structural interdependencies, it is necessary to study what new effects they induce in the dynamical behavior of the system.

The first problem we need to address is how to define dynamics on these networks. First, we can have the same dynamics in every layer so that the only difference is the connection pattern found in each layer. As we stated previously, the main characteristics of the diffusion process are controlled by the Laplacian tensor. The timescale is controlled by the second-smallest positive eigenvalue of the supra-Laplacian matrix (which is obtained by flattening the Laplacian tensor). Interestingly,

for some parameters its value is such that the diffusion process is faster in the multiplex network than it is in the separated layers (58). Conversely, it is possible to implement a different dynamical process in each layer so that they can enhance or inhibit each other, as we shall see later.

The two main diffusion processes that are studied in multilayer networks are epidemic spreading and information diffusion. In fact, sometimes the same equations are used for both processes, because in some cases they are very similar. As an example of opinion dynamics, the study by Amato et al. (59) examined the competition of opinions in a two-layer network by allowing the same individual to have a different opinion in each layer, in this way representing the possibility of individuals having different opinions depending on the context (layers represent diverse scenarios).

As noted before, epidemic-like processes have also been used to study information dissemination. We thus focus on models for disease spreading in the remainder of this section. The addition of multiple layers to the system can have dramatic effects on the outcome of an epidemic. For example, Buono et al. (60) showed that with partial overlap, i.e., with only a fraction of the nodes in one layer also being in the other layer, both the epidemic threshold and the total fraction of infected individuals significantly change even for low values of overlap.

To show the differences between studying epidemic processes on a single-layer network and a multilayer network, we use the supraadjacency matrix representation of this last system. Let us suppose that we want to study a susceptible-infected-susceptible (SIS) model in a simple network. That is, nodes can be susceptible (S) and get infected (I) with probability β when they contact infected individuals. Once a node becomes infected, it might recover with probability μ and get into the susceptible compartment again. We can express this process using a Markov chain approach such as (61)

$$p_i(t+1) = (1 - p_i(t))(1 - q_i(t)) + (1 - \mu)p(t),$$

where $p_i(t)$ is the probability of node i being infected at time t, and $q(t)$ is the probability of node i not getting infected by any neighbor, which is given by

$$q_i(t) = \prod_j [1 - \beta R_{ij} p_j(t)];$$

in this context, R_{ij} is the contact probability matrix,

$$R_{ij} = 1 - \left(1 - \frac{a_{ij}}{k_i}\right)^{\lambda}.$$

With this formulation, one can go from a contact process (one contact per unit time) when $\lambda = 1$ to a fully reactive process (all neighbors are contacted) when $\lambda \to \infty$. Note that a_{ij} stands for the element ij of the adjacency matrix. Thus, it is in this last element where the structure of the network plays a role.

This equation always has the trivial solution $p_i = 0 \forall i$. Other nontrivial solutions can be easily computed numerically by iteration. Nevertheless, it is possible to get a condition for determining whether it is possible to have nontrivial solutions or not. Indeed, linearizing q_i around 0 we get

$$\left(R - \frac{\mu}{\beta} I\right) p = 0,$$

which has nontrivial solutions if and only if $\frac{\mu}{\beta}$ is an eigenvalue of R. Thus, the epidemic threshold (i.e., the critical point) is

$$\left(\frac{\beta}{\mu}\right)_c = \frac{1}{\Lambda_{\max}},$$

where $\Lambda_{\max}$ is the largest eigenvalue of the matrix R.

Now, following Cozzo et al. (27), if we have a multiplex system, we can define the supracontact probability matrix as

$$\bar{R} = \bigoplus_{\alpha} R_{\alpha} + \frac{\gamma}{\beta} C,$$

where the R_α values are the contact probability matrices of each layer α, and C is the interlayer coupling matrix. Note that we must divide the second term of the equation by β because if we did not, once we multiply $\bar{R}$ by β, we would be spreading the disease in the same way in both intralayer links and interlayer links. That is, we would be losing the distinction between links inside layers and links that connect nodes of different layers. Likewise, γ, which represents the probability of transmitting the disease through interlayer nodes, also needs to be inside the definition of the supracontact probability matrix.

The power of this formulation is that we can use the exact same equations as in the previous case just by changing the contact probability matrix R by the supracontact probability matrix $\bar{R}$. Hence, the epidemic threshold in this framework is given by

$$\left(\frac{\beta}{\mu}\right)_c = \frac{1}{\bar{\Lambda}_{\max}}.$$

At first glance, it may seem that nothing has changed, but there is an important difference. Indeed, the largest eigenvalue value is no longer fixed by the network structure but rather depends on the ratio γ/β. Even more important, it is worth analyzing this result by means of a perturbative approach by fixing the ratio $\epsilon = \gamma/\beta \ll 1$. At first-order approximation the largest eigenvalue of $\bar{R}$ is $\bar{\Lambda}_{\max} = \max_\alpha\{\Lambda_\alpha\}$, and hence the emergence of a macroscopic steady state is determined by the layer with the largest eigenvalue. This is the dominant layer, as it sets the critical properties of the entire multilayer system.

Furthermore, de Arruda et al. (62) studied the whole spectrum of this object and found that there are multiple transitions. Specifically, for a three-layer network, the first transition changes the system from a disease-free state to an endemic one that is layer-localized; i.e., the epidemic is confined to only one layer. The second point represents the transition from a layer-localized state to a delocalized state (see **Figure 2**). We also note that a similar approach was used by Valdano et al. (63) to develop a theoretical model to investigate epidemic processes in arbitrary temporal networks.

As we have seen, the introduction of interlayer links is not straightforward and needs to be done with care as the links can represent several things. For instance, Min et al. (64) studied a model of information spreading through multiple interaction channels (layers) subject to layer-switching costs, and they demonstrated that these costs can affect the outcome in nontrivial ways so that results cannot be reproduced by using a single aggregated layer. Conversely, Zuzek et al. (65) set the transmission probability on interlayer links to 1 so that they only play the role of connecting the layers (there is no actual transmission through them).

As noted before, these kinds of models are not only interesting on their own. Indeed, multilayer networks allow for studying a plethora of more complex processes that can be naturally represented as multilayer systems. For instance, Kouvaris et al. (66) investigated pattern formation induced by diffusive transport in multiplex networks. Similarly, Biondo et al. (67) built a network composed of a trading layer and an information layer. In this way, the trading patterns of individuals, which are encoded in the first layer, are affected by the spreading of information produced in the second layer. Finally, Nicosia et al. (68) built a multilayer network mimicking neural dynamics in one layer and nutrient transport in the other. In their model, the diffusion of nutrients in one layer affects the level of synchronization in the other and is even able to produce explosive synchronization in that layer.

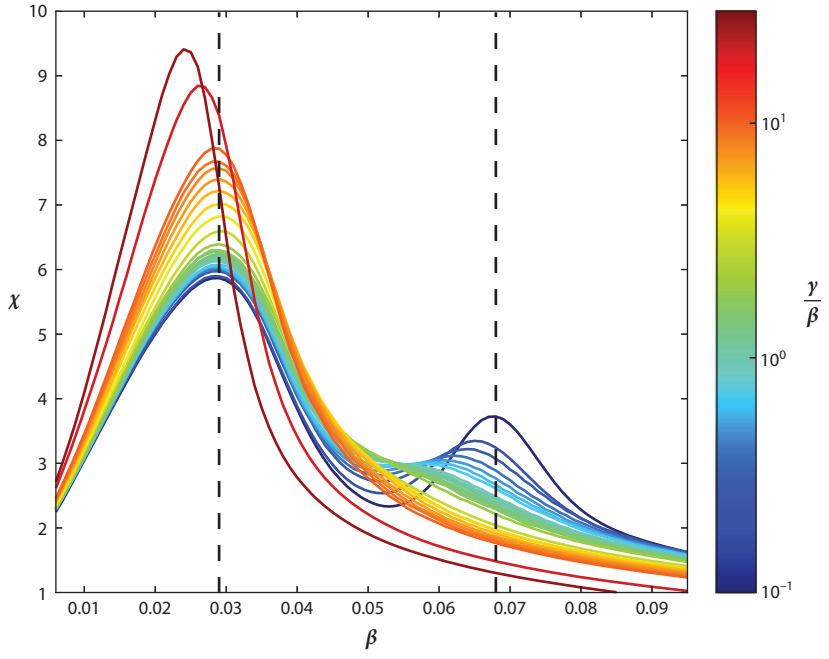

Figure 2

The figure shows the value of the susceptibility χ as a function of the probability of infection β and the value of the coupling between layers γ/β. The susceptibility shows a peak at the phase transition of the system, i.e., when the system goes from a disease-free state to an endemic one. Interestingly, for a certain set of parameters there are two phase transitions instead of one. Indeed, for low values of the infection probability and the coupling, it is possible that the disease can spread through the whole network but unable to reach other layers (first transition). If the infection probability is increased, the disease will be able to transition from said layer-localized state to a delocalized state (second transition). Figure adapted from Reference 62 with permission of the authors.

It is clear that diffusion processes can be used to study a large variety of systems, which explains their popularity. In the next section, we explore some of these systems and briefly describe how the framework of multilayer networks is being used in several fields of knowledge.

4. OTHER EXAMPLES OF MULTILAYER NETWORKS

Here, we provide a brief account of some applications in different areas in which the multilayer methodology has been used recently. We do not cover these subjects deeply, but instead we provide references to the most recent reviews on each subject so that the interested readers can easily find their way.

4.1. Multilayer Networks in Ecology

Natural systems typically exhibit multiple types of interactions, such as the same plant species interacting with both pollinators and herbivores. Thus, much can be gained by disentangling these interactions and organizing them in multilayer networks. Indeed, as Pilosof et al. (69) state, there are some multilayer layerings that can provide deep insight into the behavior and evolution of ecological systems. In particular, layers can represent the same system at different points of space

or time, which is useful when studying long-term evolution of ecological systems. If we center our attention on links, we can define each layer according to the interaction type. Conversely, if we are more interested in the organization of nodes, we can define layers by dome group identity or by the level of organization within the system.

For instance, Finn et al. (70) used the third approach to examine the role of individuals in a society of baboons. In this way, they built a multilayer network of two layers: One represents grooming while the other represents association (i.e., proximity). Then, they studied the centrality of the baboons using this representation and compared it with the results obtained for the aggregated (single-layer) network. Interestingly, they found that the most central individual in the multilayer representation is not that central in the aggregated one, and the most central individual is not central at all in the individual layers. They concluded that when studying relationships that depend on multiple interaction types, it might be necessary to use the multilayer framework to capture individual's social roles.

4.2. Multilayer Networks in Biology

Gosak et al. (71) summarized the state of network science in the study of biological systems at different scales. In addition to giving an overview of the use of single-layer networks in biological systems, they also discuss the possibilities the multilayer network formalism provides. For instance, one can build a protein–protein interaction multilayer network in which each layer represents the life stage of a bacteria (72). But this research is no longer constrained to small and simple systems. As an example, Zitnik et al. (73) built a multilayer network of molecular interactions in which each layer represents a different human tissue. By imposing a tissue hierarchy, they were able to improve the predictive power of this description compared with the single-layer case.

4.3. Multilayer Networks in Transport

Transportation systems are one example of those systems where the multilayer formulation arises in a natural way as there can be several modes of transport between two given locations. These modes can have very different properties such as velocity or carrying capacity, and thus it is important to be able to distinguish each of them when studying the whole system.

There are several ways of modeling transport systems as multilayer networks depending on the aim of the study. For instance, at a country level, one can model each transport mode, i.e., coach, trains, flights, etc., as layers and cities as nodes (74). A similar approach can be used when studying smaller systems like urban transport, in which nodes might be locations in the city and each layer might represent a mode of transport such as buses, metro, rail, etc. (75) (see **Figure 3**). However, in this last type of system, it is important to not only analyze the interplay between different modes but also between lines of each mode. Indeed, to properly take into account the transfer time from one metro line to another, for instance, it is necessary to consider each line as a single layer. Then, layers of each transport mode can be grouped into a higher-level entity, superlayers, depending on the properties one is interested in unveiling (15). Besides, as urban transports are embedded in the city, it might also be interesting to add the street pattern as a layer in order to provide novel insights into urban planning (76).

Similarly, the multilayer methodology has also been applied to study flight networks. In particular, Cardillo et al. (77) built the European air transport multilayer network in which each layer represented an airline. In this way, they showed that the resilience of the multilayer network is lower than the aggregated network. Other air transportation networks that have been analyzed using the multilayer framework include the Greek aviation network (78) and the Chinese aviation network (79, 80).

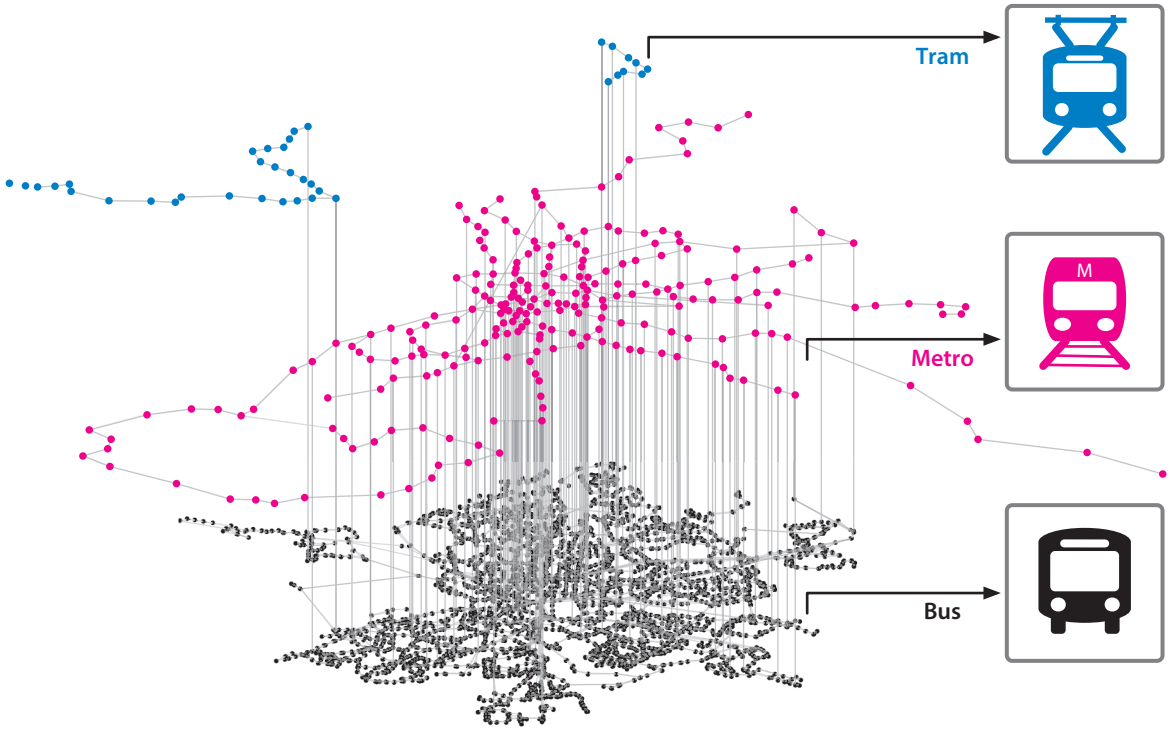

Figure 3

The multilayer representation of the transport system of Madrid. The first layer (*blue nodes*) represents the tram system, the second layer (*pink nodes*) the metro system, and the third layer (*black nodes*) the bus system. Vertical links connect stops of different transport modes that are within a 150-m radius. Figure adapted from Reference 15 with permission of the authors.

4.4. Multilayer Networks and the Human Brain

The traditional tools used to analyze the architecture of the human brain have been focused on single scales. Following Betzel et al. (81), we can distinguish three scales in brain networks, namely, the spatial scale, which refers to the granularity at which the network is defined; the temporal scale, ranging from submilliseconds to the entire lifespan; and the topological scale, ranging from individual nodes to the network as a whole. Most analysis of the human brain network done so far fix these three characteristics to a given value. However, now it is possible to address problems like the time-varying connectivity of the brain using multilayer networks. Furthermore, if one tackles the same problems, such as identifying central nodes, in single-layer brain networks and multilayer networks the results are quite different (82). Similarly, Battiston et al. (83) studied motif structures in a multilayer network composed of the anatomical and functional networks of the brain and found joint anatomofunctional motifs that differ from those found in monoplex networks.

4.5. Multilayer Networks in Economy

The multilayer formulation allows for novel approaches to the study of the dynamics of the economy. Musmeci et al. (84) applied the multilayer approach to study the structure of financial markets by building a multilayer network in which each layer represents the same data but the links are constructed using different correlation measures: Pearson, Kendall, Tail, and Partial

correlation. In this way, they can provide a complete picture of the market dependency structure, as the Pearson layer accounts for linear dependencies, the Kendall layer accounts for monotonic nonlinearity, the Tail gives information about correlations in the tails of the distributions, and finally the Partial correlation detects direct relationships that are not explained by the market.

There are several other examples of applications of the multilayer framework to study economical systems. Bargigli et al. used it to study the interbank network (85), Zeng et al. performed a multilayer network analysis of EU lobby organizations (86), Condorelli et al. studied dynamical models of bilateral trading in multilayer networks (87), Santana et al. studied investor commitment to startups and entrepreneurs (88), and Battiston et al. described financial systems as interconnected networks (89).

4.6. Multilayer Networks in Game Theory

As multilayer networks can account for several different social contexts at the same time, they are becoming quite popular in the understanding of human cooperation (90). A prominent example is interdependent network reciprocity, which can maintain healthy levels of cooperation even under extremely adverse conditions (91). However, as Battiston et al. state (92), this field is still in its infancy, and it is important not to be overly optimistic with these results. In particular, these authors showed that network interdependence can effectively promote cooperation past the limits imposed by isolated networks, but this only occurs if the multilayer network fulfills some specific conditions. Thus, the involvement in different social contexts on its own is not sufficient for the promotion of cooperation.

5. OUTLOOK

In this review, we have provided a brief and informal introduction to the subject of multilayer networks. Most of the work in this field has been done since 2012, and yet an exhaustive review will likely produce several hundreds of pages. This illustrates the potential of the new methodology to tackle real-world problems. Here, we have introduced the reader to a few such problems, but the list of possible applications is still growing. We would like to mention just a few of the open challenges and promising venues for the use of the multilayer formalism to round off this review.

First, there are still some fundamental problems to address regarding the theoretical description of multilayer networks. The majority of work conducted to date has dealt with the simplest networks one can consider, that is, undirected and unweighted networks in each layer. Although several results can be shown to hold for the case of weighted multilayer networks, this is not so when one deals with directed networks. In this latter scenario, the adjacency or supraadjacency matrices are not symmetric, which makes studying them more difficult, for instance, in terms of the spectral properties of the system. Second, there is still much to do regarding adaptive and temporal multilayer networks. These latter cases could describe systems as relevant as the brain, social networks in which new connections are created and old ones are deleted as time goes on, or even when one studies dynamics in which nodes can react and adapt in many ways (which, for instance, might require working with diffusion coefficients that are time dependent).

As for new applications, the use of multilayer networks to study complex biological systems is a promising direction. At the cellular level, proteins and genes are often involved in many signaling, transcriptional, or metabolic regulations—which in turn depend on external stimulus or the environmental conditions—that take place concurrently, which calls for using multiple channels to describe each of these pathways, that is, a multilayer framework. At the population level, a field that could greatly benefit from the new formalism is that of disease spreading. This

is the case for an increasing number of real scenarios in which two or more diseases propagate on top of the same host population, giving rise to complex dynamical interdependencies like cross-immunization or cooperation between the diseases, as is the case with tuberculosis and HIV, that can be accounted for by considering multiple networks of contacts. Despite the many advances of the past several years in multilayer networks, we expect that the field will continue to grow in terms of both foundational research as well as their application to several problems in many different areas of research, from the social and biological sciences to natural and technological systems.

DISCLOSURE STATEMENT

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